Computational Tools for Comparing Asymmetric GARCH Models via Bayes Factors

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GARCH Model

The GARCH(p,q) model,

$$
\begin{aligned}
y_t &= \epsilon_t \sqrt{h_t}\ ,\ \epsilon_t \sim D(0,1) \\
h_t &= \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i}.
\end{aligned}
$$

h_t: conditional variance of y_t given $\{y_{t-1}, y_{t-2}, \ldots\}$

ϵ_t: i.i.d. errors

$D(0,1)$: denotes a distribution with mean zero and variance 1

$\omega > 0$

$\alpha_i \geq 0$, $i = 1, \ldots, p$

$\beta_i \geq 0$, $i = 1, \ldots, q$

$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i < 1$

The conditional likelihood function of the model,

$$l(\boldsymbol{\theta}) = \prod_{t=s+1}^{n} h_t^{-1/2} p_\epsilon(y_t/\sqrt{h_t}), \; s = \max(p, q),$$

$$\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_{p+q}) = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'.$$

There are a number of proposals in the literature to introduce skewness in unimodal symmetric distributions. In particular, Fernandez and Steel (1998) presented a general method for transforming any continuous unimodal and symmetric distribution into a skewed one by changing the scale at each side of the mode. They proposed the following class of skewed distributions indexed by a shape parameter $\gamma \in (0, \infty)$, which describes the degree of asymmetry,

$$s(x|\gamma) = \frac{2}{\gamma + 1/\gamma} \left\{ f\left(\frac{x}{\gamma}\right) I_{[0,\infty)}(x) + f(\gamma x) I_{(-\infty,0)}(x) \right\}, \gamma > 0.$$

with $s(x|\gamma = 1) = f(x)$.

We compare the following different distributions for the error term: symmetric normal distribution, standardized Student-t distribution, the generalized error distribution and their skewed versions.

	error dist.	parameters
symmetric	stardard normal	$\boldsymbol{\theta}$
	standardized t	$\boldsymbol{\theta} = (\boldsymbol{\theta}, \nu),\ \nu > 2$
	standardized GED	$\boldsymbol{\theta} = (\boldsymbol{\theta}, \nu),\ \nu \in (0, 2)$
asymmetric	skew normal	$\boldsymbol{\theta} = (\boldsymbol{\theta}, \gamma)$
	skew t	$\boldsymbol{\theta} = (\boldsymbol{\theta}, \nu, \gamma),\ \nu > 2,\ \gamma > 0$
	skew GED	$\boldsymbol{\theta} = (\boldsymbol{\theta}, \nu, \gamma),\ \nu \in (0, 2),\ \gamma > 0$

Figure 1: Density functions of the standard normal, standardized Student-t with 5 degrees of freedom, Laplace and GED with $\nu = 1.5$

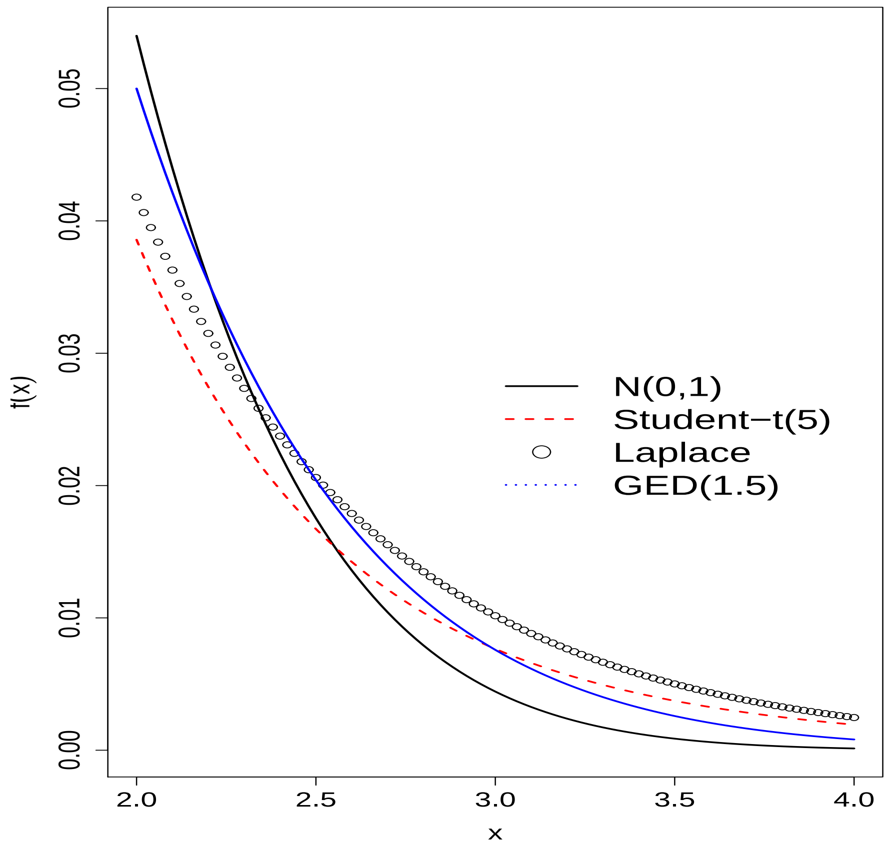
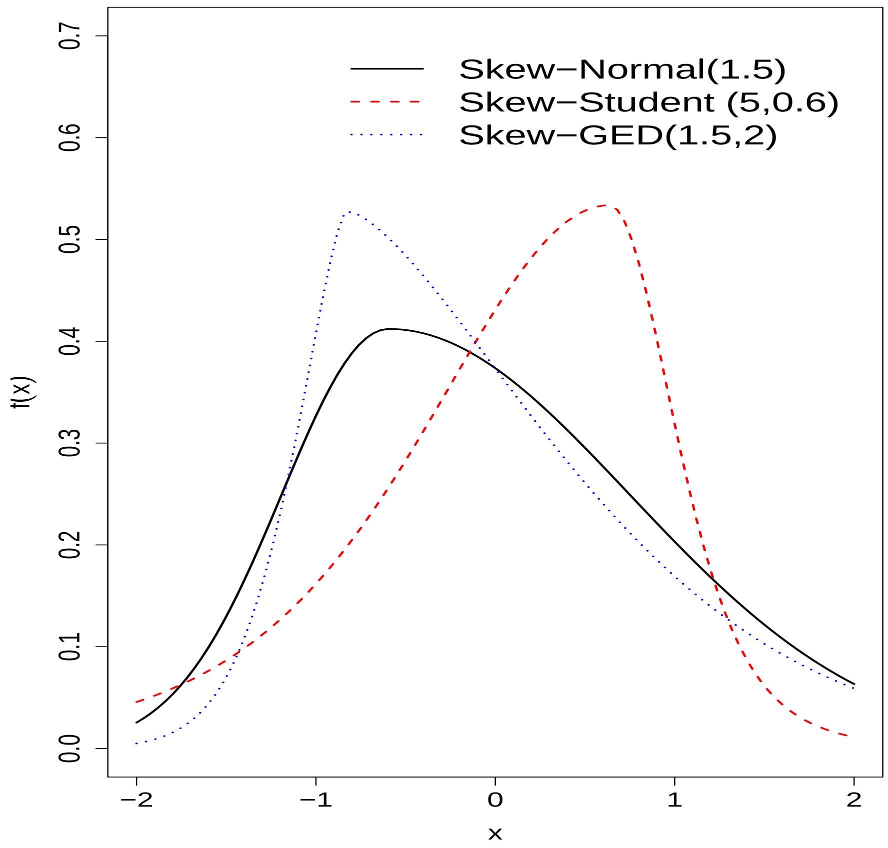


Figure 2: Density functions of the skew normal, skew Student-t and skew GED.



Priors Distributions for GARCH Model

$$\phi_j = \log\left(\frac{\theta_j}{1-\theta_j}\right) \sim N(0, \sigma_\phi^2), \quad j = 1, \ldots, p+q$$

$$\phi_0 = \log\left(\frac{\theta_0}{\bar{y}^2 - \theta_0}\right) \quad \text{where} \quad \bar{y}^2 = (1/n)\sum y_t^2.$$

Tails thickness parameter for the GED: $\psi = \log\left(\frac{\nu}{2-\nu}\right) \sim N(0, \sigma_\psi^2)$.

Degrees of freedom for the Student-t: $\nu \sim \text{Exponential}(\beta)$, $\beta = 0.1$.

Skewness parameter for the Skew normal, skew t and skew GED: $\gamma \sim N(0, 0.64^{-1})$ truncated to $\gamma > 0$, in which case,

$$Var(\gamma) \approx 0.57, \quad E(\gamma) \approx 1 \text{ and } P(0 < \gamma < 1) \approx 0.58.$$

Prior is centered around the symmetric version of the distribution and gives approximately equal weights to left and right skewness.

Random Walk Metropolis for the GARCH

1. Set initial values for the GARCH parameters $\boldsymbol{\theta}^{(0)}$ and transform to $\boldsymbol{\phi}^{(0)} \in \mathbb{R}^d$.

2. At iteration j, generate a vector from the random walk kernel,

$$\boldsymbol{\phi}' = \boldsymbol{\phi}^{(j-1)} + \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim N(\mathbf{0}, \tau\boldsymbol{\Sigma})$$

3. If $\sum_{j=1}^{p+q} e^{\phi'_j}/(1 + e^{\phi'_j}) < 1$ set $\boldsymbol{\phi}^{(j)} = \boldsymbol{\phi}'$ with probability

$$\alpha(\boldsymbol{\phi}, \boldsymbol{\phi}') = \min\left\{1, \frac{l(\boldsymbol{\phi}')p(\boldsymbol{\phi}')}{l(\boldsymbol{\phi})p(\boldsymbol{\phi})}\right\},$$

   otherwise reject the move and set $\boldsymbol{\phi}^{(j)} = \boldsymbol{\phi}^{(j-1)}$.

4. Repeat until convergence.

In step 2 above, τ is a constant to tune the acceptance rate and Σ is estimated from the approximate Hessian matrix of the target density evaluated at its mode.

Approximating the marginal likelihood

Chib and Jeliazkov (2001)

$$p(\boldsymbol{y}|M_i) = \frac{p(\boldsymbol{y}|\boldsymbol{\theta}_i^*, M_i)p(\boldsymbol{\theta}_i^*|M_i)}{\pi(\boldsymbol{\theta}_i^*|\boldsymbol{y}, M_i)}$$

$$\alpha(\boldsymbol{\theta}, \boldsymbol{\phi})q(\boldsymbol{\phi}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}|\boldsymbol{y}) = \alpha(\boldsymbol{\theta}^*, \boldsymbol{\theta})q(\boldsymbol{\theta}|\boldsymbol{\theta}^*)\pi(\boldsymbol{\theta}^*|\boldsymbol{y})$$

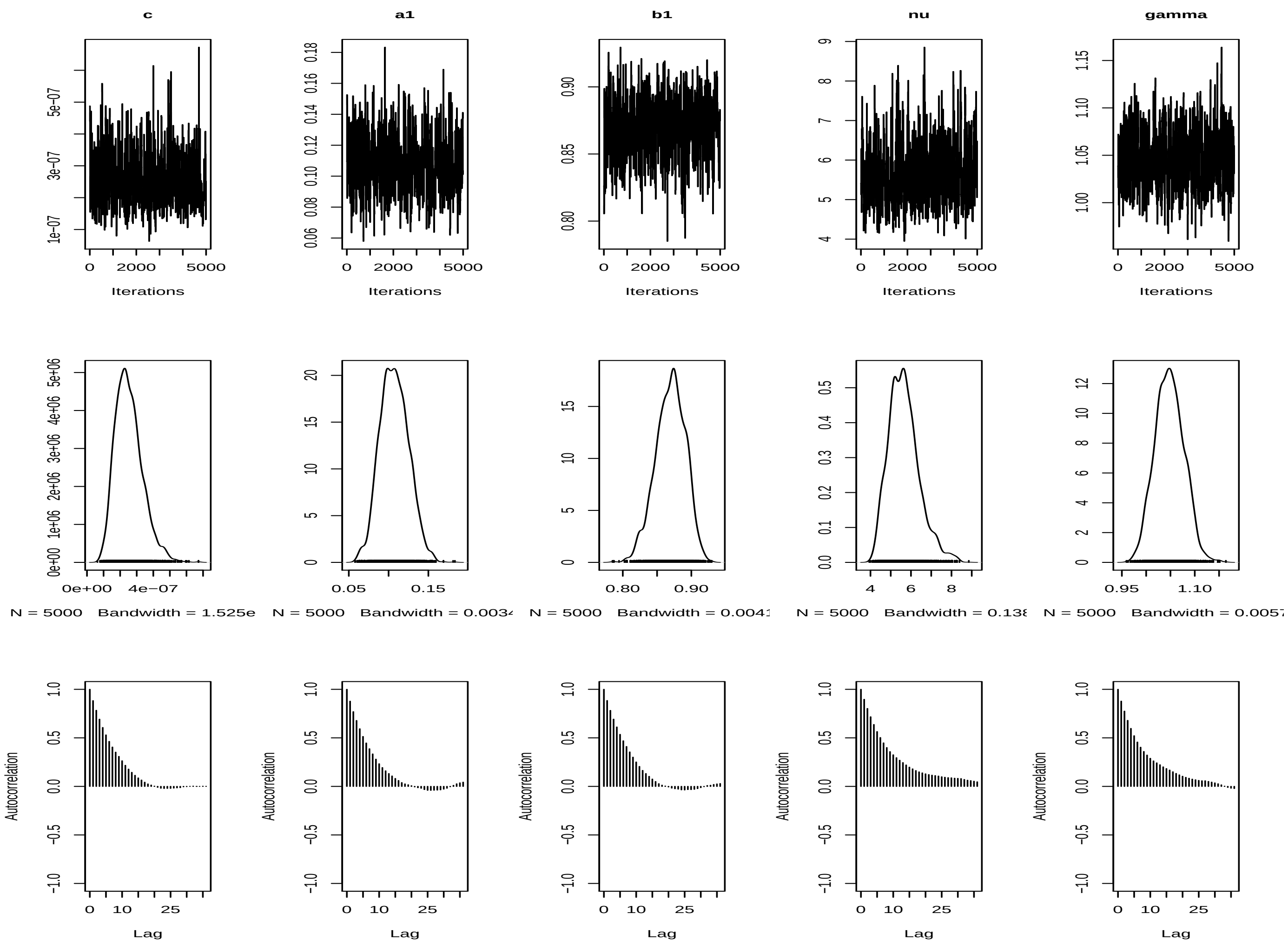
$$\Downarrow$$

$$\pi(\boldsymbol{\theta}^*|\boldsymbol{y}) = \frac{\displaystyle\int \alpha(\boldsymbol{\theta}, \boldsymbol{\theta}^*)q(\boldsymbol{\theta}^*|\boldsymbol{\theta})\pi(\boldsymbol{\theta}|\boldsymbol{y})d\boldsymbol{\theta}}{\displaystyle\int \alpha(\boldsymbol{\theta}^*, \boldsymbol{\theta})q(\boldsymbol{\theta}|\boldsymbol{\theta}^*)d\boldsymbol{\theta}} \approx \frac{N^{-1}\sum_{g=1}^{N} \alpha(\boldsymbol{\theta}^{(g)}, \boldsymbol{\theta}^*)q(\boldsymbol{\theta}^*|\boldsymbol{\theta}^{(g)})}{J^{-1}\sum_{j=1}^{J} \alpha(\boldsymbol{\theta}^*, \boldsymbol{\theta}^{(j)})}$$

Table 1: Daily Exchange Rates of various currencies relative to the US dollar. Estimates of posterior model probabilidades.

	normal	t	GED	skew normal	skew t	skew GED
Australian Dollar	0.0000	0.0183	0.0123	0.0000	0.0586	0.9108
British Pound	0.0000	0.6283	0.0007	0.0000	0.3707	0.0003
Canadian Dollar	0.0000	0.1335	0.0002	0.0000	0.8655	0.0008
French Franc	0.0000	0.2376	0.0001	0.0000	0.7623	0.0001
German Marc	0.0000	0.2648	0.0000	0.0000	0.7351	0.0001
Japanese Yen	0.0000	0.0801	0.0000	0.0000	0.9198	0.0000

Figure 3: MCMC paths, density estimates and autocorrelations for the GARCH(1,1) with skew Student errors (Canadian Dollar exchage rates).



Concluding Remarks

- The main contribution was to provide computational tools for estimating and comparing GARCH models using MCMC methods and an approximation to the Bayes factor.
- Other recent approaches to model comparison via marginal likelihood estimation are currently under investigation.
- The methods developed in Friel and Pettit (2008), Chen (2005), Chib and Jeliazvok (2005) are very promissing and we seek to adapt them to the context of models in the GARCH family.
- For real time series we found evidence in favour of skewed distributions for the error term. Similar findings are reported by other authors (see for example Cappuccio et al. 2004).

References

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