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INLA - Introduction

Elias T. Krainski

Outline

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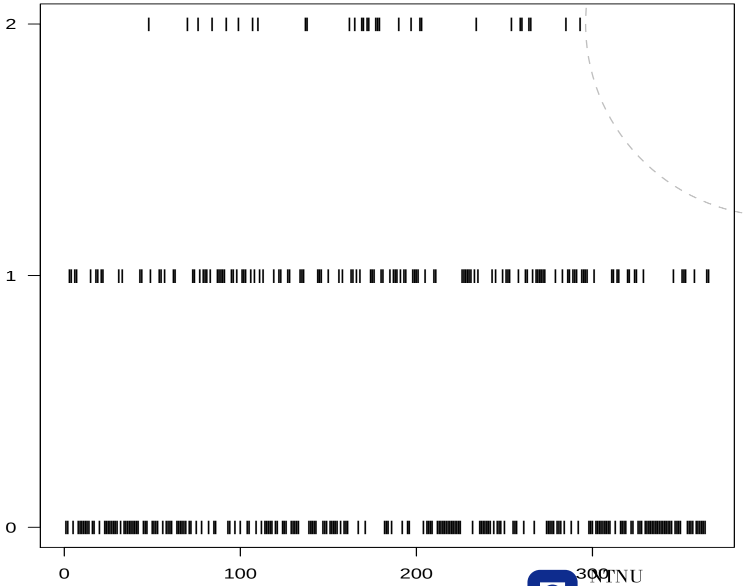
Number of raining days in Tokyo, for each yearly day in two years



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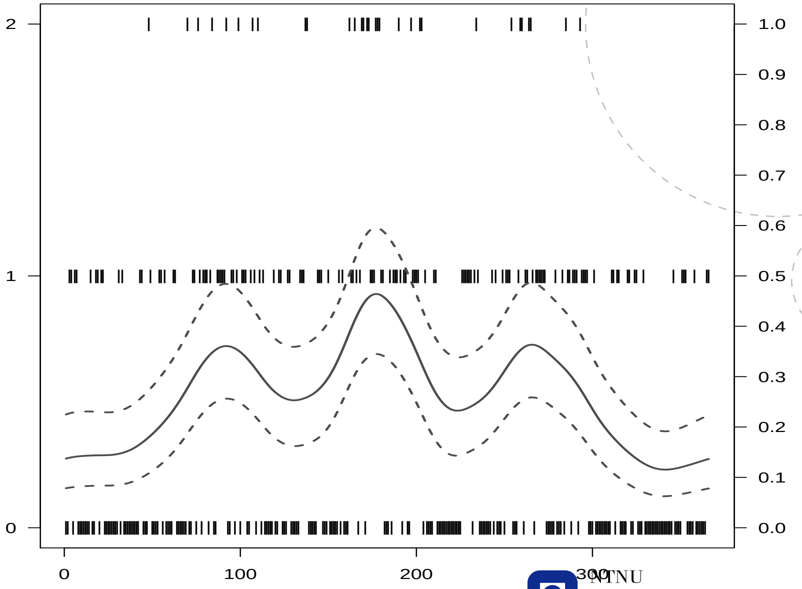

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A model for Tokyo data

Observation model

$$y_i \sim \text{Binomial}(n_i, p_i),$$

for $i = 1, 2, ..., 366$

$$n_i = \begin{cases} 1, & \text{for 29 February} \\ 2, & \text{other days} \end{cases} \tag{1}$$

$$y_i \in \begin{cases} \{0, 1\}, & \text{for 29 February} \\ \{0, 1, 2\}, & \text{other days} \end{cases} \tag{2}$$

$$p_i = \frac{1}{1 + exp(-x_i)}$$

probability on day i depends on x_i

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Smoothing $\boldsymbol{x}$

— Let

$$x_i|\boldsymbol{x}_{-i} \sim N(\overline{x}_i, \frac{\sigma^2}{2})$$

where

$$\overline{x}_i = \begin{cases} \frac{x_2+x_{366}}{2} & \text{if} \quad i = 1 \\ \frac{x_{i-1}+x_{i+1}}{2} & \text{if} \quad 1 < i < 366 \\ \frac{x_{365}+x_1}{2} & \text{if} \quad i = 366 \end{cases} . \quad (3)$$

and $\theta = 1/\sigma^2$

— θ is controls the variation of $\boldsymbol{x}$

- so, related to variantion of p_i

— as $\theta > 0$: people usually use $\pi(\theta) \sim$ Gamma(a, b)

Outline

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Bayesian hierarchical model

- $\boldsymbol{y}$: observed response data
- θ_1: likelihood parameter(s)

$$\boldsymbol{y}|\boldsymbol{x},\theta_1 \sim \pi(\boldsymbol{y}|\boldsymbol{x},\theta_1) = \prod_{i=1}^{n} \pi(y_i|x,\boldsymbol{\theta_1}) \qquad \text{(ind. cond.)}$$

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- $\boldsymbol{x}$: latent/unobserved field
  - Gaussian $\rightarrow$ to use INLA
- θ_2: latent field parameter(s)

$$\boldsymbol{x}|\theta_2 \sim \pi(\boldsymbol{x}|\theta_2) = N(\boldsymbol{0}, \boldsymbol{Q}(\theta_2)^{-1})$$

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Bayesian hierarchical model

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- Gaussian $\rightarrow$ to use INLA

— θ_2: latent field parameter(s)

$$\boldsymbol{x}|\theta_2 \sim \pi(\boldsymbol{x}|\theta_2) = N(\boldsymbol{0}, \boldsymbol{Q}(\theta_2)^{-1})$$

— in short: $\boldsymbol{\theta} = \{\theta_1, \theta_2\}$ (hyperparameter)

$$\boldsymbol{\theta} \sim \pi(\boldsymbol{\theta}) \rightarrow \text{to be Bayesian}$$

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$\pi(\boldsymbol{y}|\boldsymbol{x},\theta)$: likelihood

Depends on

— which kind of data values we have

- binary (yes/no response, binary image)
- counts (people infected with a disease in each area)
- continuous - or + (stock return, temperature)
- continuous + (rainfall amount, fish weight)
- survival (recovery time, time to death)

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- continuous + (rainfall amount, fish weight)
- survival (recovery time, time to death)

— the way it is collected

- **usual**: each observational unit gives one value
- each observational unit gives more than one value
- point process: locations (time, spatial) of events

$\pi(\boldsymbol{x}|\boldsymbol{Q}(\theta))$: The latent field prior

— It is

- unobserved
- called Gaussian latent random field
- the most important ingredient in INLA

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- covariate coefficients
- unobserved effects

$\pi(\boldsymbol{x}|\boldsymbol{Q}(\theta))$: The latent field prior

- It is
  - unobserved
  - called Gaussian latent random field
  - the most important ingredient in INLA
- it represents
  - covariate coefficients
  - unobserved effects
- it can be
  - unstructured (Tokyo: p_i doesn't depends p_j)
  - structured (Tokyo: p_i depends on neighbour days)
  - more than one (structured(s) + unstructured(s) + covariate(s))

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$\pi(\theta)$: The θ prior

- parameters from the likelihood and $\boldsymbol{x}$ distribution
- examples (likelihood):
  - precision parameter of the Gaussian or Gamma
  - dispersion parameter in Beta, negative binomial
  - zero-inflation probability

$\pi(\theta)$: The θ prior

- parameters from the likelihood and $\boldsymbol{x}$ distribution
- examples (likelihood):
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  - dispersion parameter in Beta, negative binomial
  - zero-inflation probability
- examples (latent field)
  - precision parameter in, usually, all of those
  - correlation parameter (in some for time series modelling)
  - range parameter (in some for spatial data modelling)

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Outline

Tokyo example

Hierarchical model

On the Tokyo model

Heart model example

Bayesian inference

INLA overview

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A model for Tokyo data

Observation model

$$y_i \sim \text{Binomial}(n_i, p_i)$$

$$p_i = \frac{1}{1 + \exp(-x_i)}$$

the likelihood has no θ

$$\pi(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{366} \pi(y_i|x_i)$$

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Latent model

$$\pi(\boldsymbol{x}|\boldsymbol{\theta}) \propto \exp\left\{-\frac{\theta}{2}\left[(x_1 - x_{366})^2 + \sum_{i=2}^{366}(x_i - x_{i-1})^2\right]\right\} \quad (4)$$

$$= \exp\left\{-\frac{\theta}{2}\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}\right\} \quad (5)$$

Latent model

$$\pi(\boldsymbol{x}|\boldsymbol{\theta}) \propto \exp\left\{-\frac{\theta}{2}\left[(x_1 - x_{366})^2 + \sum_{i=2}^{366}(x_i - x_{i-1})^2\right]\right\} \quad (4)$$

$$= \exp\left\{-\frac{\theta}{2}\boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x}\right\} \quad (5)$$

where $\boldsymbol{R} = \begin{pmatrix} 2 & -1 & & & & & -1 \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ -1 & & & & & -1 & 2 \end{pmatrix}$

$\boldsymbol{Q}(\theta) = \theta\boldsymbol{R}$

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Latent model **warning**

$$\exp\left\{-\frac{\theta}{2}\left[(x_1 - x_{366})^2 + \sum_{i=2}^{366}(x_i - x_{i-1})^2\right]\right\} \quad (6)$$

$$(7)$$

intrinsic/improper

$$\begin{aligned} x_i = 20, \quad x_{i-1} = 10 \quad &\rightarrow \quad x_i - x_{i-1} = 10 \\ x_i = 10020, \quad x_{i-1} = 10010 \quad &\rightarrow \quad x_i - x_{i-1} = 10 \end{aligned}$$

constraint or take the intercept out

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$\pi(\theta)$ problem

- Tokyo example: $Q(\boldsymbol{\theta}) = \theta \boldsymbol{R}$
  - bigger θ less variation of $\boldsymbol{x}$
    - related to the variation of p_i
- $\theta > 0$: people usually use $\theta \sim \text{Gamma}(a, b)$
- improper distribution: θ values depends on $\boldsymbol{R}$
  - hard to interpret θ (a=?????, b=?????)

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$\pi(\boldsymbol{x}|\theta = 1)$ and n

The marginal variance and n relation

```
rw.var <- function(n, order) {
    R <- as.matrix(INLA::inla.rw(n, order=order))
    mean(diag(INLA::inla.ginv(R, rankdef=order)))
}
n <- c(10, 100, 366, 1000); names(n) <- n
rbind(rw1=sapply(n, rw.var, order=1),
      rw2=sapply(n, rw.var, order=2))

##       10      100         366         1000
## rw1 1.65   16.665    60.99954     166.6665
## rw2 2.40 2381.190 116733.95702 2380955.1304
```

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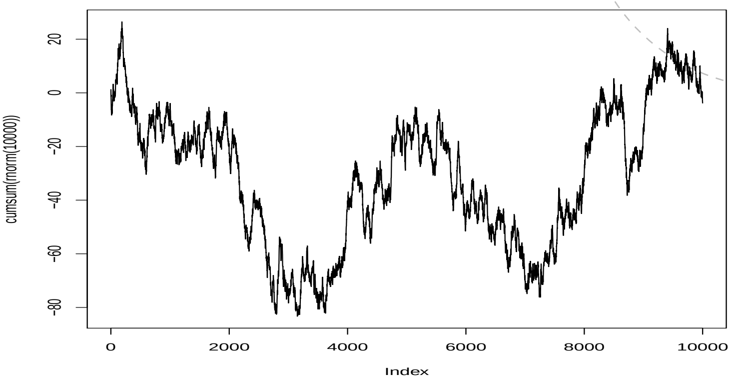
$\pi(\boldsymbol{x}|\theta = 1)$: one realization



We need to control the marginal variance!

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$\pi(\theta)$ solution

1. scale the model $\rightarrow$ easy to interpret θ
   - Tutorial on `scale.option` at `www.r-inla.org/`

$\pi(\theta)$ solution

1. scale the model $\rightarrow$ easy to interpret θ
   - Tutorial on `scale.option` at `www.r-inla.org/`
2. AND (new idea) Penalized complexity prior
   - P0: basic model: $p_i = p_0$
   - P1: complex model: p_i varies
   - Kullback-Leibler divergence (KLD)
     - a distance from P1 model to P0, KLD(P0/P0) = 0
   - allow variation on p_i
   - AND supports the basic model
     - Gamma(a, b) always overfits

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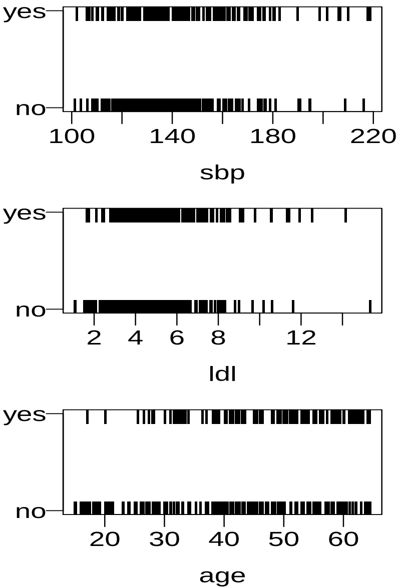

Heart data, from `catdata` package

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Model?

- Generalized Linear Model?
  - linear predictor
  - nonlinear link (logit, probit, and others)
- Nonlinear effect from covariate?
  - parametric nonlinear function?
  - non parametric nonlinear function?
- Bayesian?

```
l <- glm(y ~ age, family=binomial, data=heart)
b <- inla(y ~ age, family='binomial', data=heart)
round(cbind(l=coef(summary(l))[,1:2], b=b$summary.fix[,1:2])

##             l.Estimate l.Std. Error  b.mean   b.sd
## (Intercept)    -3.5217       0.4160 -3.5257 0.4160
## age             0.0641       0.0085  0.0643 0.0085
```

GxxMs: Different names for the same thing

GLMM/GAM/GAMM/+++

- Perhaps the most important class of statistical models
- Many “different” models belong to this class
- No good (enough) MCMC solution around
- Even frequentist approaches does not scale well computationally

Back to linear models

Consider the linear model

$$y_i = \beta_0 \boldsymbol{F}_{i1} + \beta_1 \boldsymbol{F}_{i2} + \beta_2 \boldsymbol{F}_{i3} + u_i + \epsilon_i$$

where $\boldsymbol{F}$ is the design matrix (with ones at first column)

- y_i is an observation
- μ is the intercept
- β_0, β_1 and β_2 are the regression coefficients
- $\boldsymbol{u}$ is a random effect
- ϵ_i is i.i.d. normal observation noise.

How it works in a Bayesian framework?

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Bayesian linear models

Linear model:

$$y_i = \beta_0 \boldsymbol{F}_{i1} + \beta_1 \boldsymbol{F} i2 + \beta_2 \boldsymbol{F}_{i1} + u_i + \epsilon_i$$

Bayesian model: chose priors. Usual choices:

- $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)^T \sim N(\mathbf{0}, \tau_{\text{fix}}^{-1} \boldsymbol{I})$, where τ_{fix} is a small number
- $\boldsymbol{u} \sim N(\mathbf{0}, \boldsymbol{Q}_{\boldsymbol{u}}^{-1})$ where the *precision matrix* $\boldsymbol{Q}$ is known
- $\epsilon \sim N(\mathbf{0}, \tau_{\text{n}}^{-1} \boldsymbol{I})$

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What does this look like? (Horror slide!)

$(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\beta})$ are jointly Gaussian!

$$\pi(\boldsymbol{y}|\boldsymbol{u}, \boldsymbol{\beta}) \propto \exp\left(-\frac{\tau_\text{n}}{2}(\boldsymbol{y}-\boldsymbol{u}-\boldsymbol{F}^T\boldsymbol{\beta})^T(\boldsymbol{y}-\boldsymbol{u}-\boldsymbol{F}^T\boldsymbol{\beta})\right)$$
$$= \exp\left(-\frac{\tau_\text{n}}{2}\begin{pmatrix}\boldsymbol{y}^T & \boldsymbol{u}^T & \boldsymbol{\beta}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{I} & -\boldsymbol{I} & -\boldsymbol{F}^T \\ -\boldsymbol{I} & \boldsymbol{I} & \boldsymbol{F}^T \\ -\boldsymbol{F} & \boldsymbol{F} & \boldsymbol{F}^T\boldsymbol{F}\end{pmatrix}\begin{pmatrix}\boldsymbol{y} \\ \boldsymbol{u} \\ \boldsymbol{\beta}\end{pmatrix}\right)$$

It follows that

$$\pi(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\beta}) = \pi(\boldsymbol{y}|\boldsymbol{u}, \boldsymbol{\beta})\pi(\boldsymbol{u})\pi(\boldsymbol{\beta})$$
$$\propto \exp\left(-\frac{\tau_\text{n}}{2}\begin{pmatrix}\boldsymbol{y}^T & \boldsymbol{u}^T & \boldsymbol{\beta}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{I} & -\boldsymbol{I} & -\boldsymbol{F}^T \\ -\boldsymbol{I} & \boldsymbol{I}+\tau_\text{n}^{-1}\boldsymbol{Q}_{\boldsymbol{u}} & \boldsymbol{F}^T \\ -\boldsymbol{F} & \boldsymbol{F} & \boldsymbol{F}^T\boldsymbol{F}+\frac{\tau_\text{fix}}{\tau_\text{n}}\boldsymbol{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{y} \\ \boldsymbol{u} \\ \boldsymbol{\beta}\end{pmatrix}\right)$$

How can we use this?

From multivariate Gaussian distributions:
If

$$\boldsymbol{x} \equiv \begin{pmatrix} \boldsymbol{x}_A \\ \boldsymbol{x}_B \end{pmatrix} \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_A \\ \boldsymbol{\mu}_B \end{pmatrix}, \begin{pmatrix} \boldsymbol{Q}_{AA} & \boldsymbol{Q}_{AB} \\ \boldsymbol{Q}_{BA} & \boldsymbol{Q}_{BB} \end{pmatrix}^{-1} \right),$$

then the conditional distribution is given by

$$\boldsymbol{x}_A | \boldsymbol{x}_B \sim N\left(\boldsymbol{\mu}_A - \boldsymbol{Q}_{AA}^{-1} \boldsymbol{Q}_{AB} (\boldsymbol{x}_B - \boldsymbol{\mu}_B), \boldsymbol{Q}_{AA}^{-1} \right).$$

We can easily compute the marginal distributions for $u_i | \boldsymbol{y}$ and $\beta_i | \boldsymbol{y}$.

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Non Gaussian likelihood?

General framework:

— include the linear predictor in $\boldsymbol{x}$

- with a small fixed variance

—

$$\pi(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{\#\text{data}} \pi(y_i|x_i)$$

— Gaussian approximantion does the rest

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Further Examples

— Dynamic linear models
— Stochastic volatility models (famously difficult with MCMC)
— Generalised linear (mixed) models
— Generalised additive (mixed) models
— Spline smoothing
— Semiparametric regression
— Space-varying (semiparametric) regression models
— Disease mapping
— Log-Gaussian Cox-processes
— Model-based geostatistics (*)
— Spatio-temporal models
— Survival analysis
— +++

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Outline

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On our Bayesian hierarchical model

— Inference on (what we know about) θ and $\boldsymbol{x}$ given $\boldsymbol{y}$
- in maths: $\pi(\boldsymbol{x}|\boldsymbol{y})$ and $\pi(\theta|\boldsymbol{y})$

— considering $\pi(\boldsymbol{y}|\boldsymbol{x},\theta)$, $\pi(\boldsymbol{x}|\theta)$ and $\pi(\theta)$

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On our Bayesian hierarchical model

- Inference on (what we know about) $\boldsymbol{\theta}$ and $\boldsymbol{x}$ given $\boldsymbol{y}$
  - in maths: $\pi(\boldsymbol{x}|\boldsymbol{y})$ and $\pi(\boldsymbol{\theta}|\boldsymbol{y})$
- considering $\pi(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta})$, $\pi(\boldsymbol{x}|\boldsymbol{\theta})$ and $\pi(\boldsymbol{\theta})$
- using the Bayes theorem,

$$\pi(\boldsymbol{x}|\boldsymbol{y}) = \int \pi(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) = \int \pi(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{x}$$

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$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) = \int \pi(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{x}$$

- even more...
  - $\pi(\theta_j|\boldsymbol{y})$, $j = 1, ..., \dim(\boldsymbol{\theta})$
  - $\pi(x_i|\boldsymbol{y})$, $i = 1, ..., \dim(\boldsymbol{x})$

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The inference problem

— we have to compute

$$\pi(x_i|\boldsymbol{y}) \propto \int_{x_{\{-i\}}} \int_{\boldsymbol{\theta}} \pi(y|\boldsymbol{x}, \theta)\pi(\boldsymbol{x}|\theta)\pi(\theta)\, d\theta\, d\boldsymbol{x}_{\{-i\}}$$

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and

$$\pi(\theta_j|\boldsymbol{y}) \propto \int_{x} \int_{\boldsymbol{\theta}_{\{-j\}}} \pi(y|\boldsymbol{x},\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}_{\{-j\}}\, d\boldsymbol{x}$$

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and

$$\pi(\theta_j|\boldsymbol{y}) \propto \int_{x} \int_{\boldsymbol{\theta}_{\{-j\}}} \pi(y|\boldsymbol{x},\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}_{\{-j\}}\, d\boldsymbol{x}$$

— remember

- dim($\boldsymbol{\theta}$) is small
- dim($\boldsymbol{x}$) is not small
- we have to compute very high dimensional integrals

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The inference problem

- we have to compute

$$\pi(x_i|\boldsymbol{y}) \propto \int_{x_{\{-i\}}} \int_{\boldsymbol{\theta}} \pi(y|\boldsymbol{x},\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}\, d\boldsymbol{x}_{\{-i\}}$$

  and

$$\pi(\theta_j|\boldsymbol{y}) \propto \int_{x} \int_{\boldsymbol{\theta}_{\{-j\}}} \pi(y|\boldsymbol{x},\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}_{\{-j\}}\, d\boldsymbol{x}$$

- remember
  - dim($\boldsymbol{\theta}$) is small
  - dim($\boldsymbol{x}$) is not small
  - we have to compute very high dimensional integrals
- typically they are not analytically tractable
  - $\rightarrow$ we have to approach

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using MCMC

— single-site: compute (the expressions) for

- $p(\theta_j|\boldsymbol{\theta}_{-j}, \boldsymbol{x}, \boldsymbol{y})$
- $p(x_i|\boldsymbol{x}_{-i}, \boldsymbol{\theta}, \boldsymbol{y})$

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using MCMC

— single-site: compute (the expressions) for
  - $p(\theta_j|\boldsymbol{\theta}_{-j}, \boldsymbol{x}, \boldsymbol{y})$
  - $p(x_i|\boldsymbol{x}_{-i}, \boldsymbol{\theta}, \boldsymbol{y})$

— draw samples from such conditionals
  - WinBUGS, OpenBUGS, JAGS, and others

— use these samples to summarize $p(\boldsymbol{x})$ and $p(\boldsymbol{\theta})$

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using MCMC

- single-site: compute (the expressions) for
  - $p(\theta_j | \boldsymbol{\theta}_{-j}, \boldsymbol{x}, \boldsymbol{y})$
  - $p(x_i | \boldsymbol{x}_{-i}, \boldsymbol{\theta}, \boldsymbol{y})$
- draw samples from such conditionals
  - WinBUGS, OpenBUGS, JAGS, and others
- use these samples to summarize $p(\boldsymbol{x})$ and $p(\boldsymbol{\theta})$
- warning
  - sampling from $x_i | \boldsymbol{x}_{-i}, \boldsymbol{\theta}, y$
    - slow convergence when strong dependence
    - does not works for our example...
  - better: draw joint sample from $\boldsymbol{x} | \boldsymbol{\theta}, y$
  - best: use INLA

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Outline

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What INLA does

- INLA does:
  - compute marginals of $\pi(x_i|\boldsymbol{y})$ and $\pi(\theta_j|\boldsymbol{y})$
- how?
  - approach $\pi(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})$ to approach $\pi(\boldsymbol{\theta}|\boldsymbol{y})$
  - explore $\pi(\boldsymbol{\theta}|\boldsymbol{y})$
    - approach $\pi(\theta_j|\boldsymbol{y})$
  - approach $\pi(x_i|\boldsymbol{x}_{-i})$

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Important ingredient

The GMRF-approximation

$$\pi(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{y}) \quad \propto \quad \exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \sum_i \log \pi(y_i|x_i)\right)$$

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Important ingredient

The GMRF-approximation

$$\begin{aligned}\pi(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{y}) &\propto \exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \sum_i \log\pi(y_i|x_i)\right)\\ &\approx \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T(\boldsymbol{Q}+\text{diag}(\boldsymbol{c}))(\boldsymbol{x}-\boldsymbol{\mu})\right)\\ &= \pi_G(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})\end{aligned}$$

$c_i = -\frac{dl_i^2}{dx_i^2}$ where $l_i = \log(\pi(y_i|x_i))$, $i = 1, ..., \#$ data

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Important ingredient

The GMRF-approximation

$$
\begin{aligned}
\pi(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{y}) &\propto \exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \sum_i \log \pi(y_i|x_i)\right) \\
&\approx \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T(\boldsymbol{Q} + \text{diag}(\boldsymbol{c}))(\boldsymbol{x}-\boldsymbol{\mu})\right) \\
&= \pi_G(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})
\end{aligned}
$$

$c_i = -\frac{dl_i^2}{dx_i^2}$ where $l_i = \log(\pi(y_i|x_i))$, $i = 1, ..., \#$ data

— Markov and computational properties (on $\boldsymbol{Q}$) are preserved

Important ingredient

The GMRF-approximation

$$\begin{aligned}\pi(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{y}) &\propto \exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \sum_i \log \pi(y_i|x_i)\right) \\ &\approx \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T(\boldsymbol{Q}+\text{diag}(\boldsymbol{c}))(\boldsymbol{x}-\boldsymbol{\mu})\right) \\ &= \pi_G(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})\end{aligned}$$

$c_i = -\frac{dl_i^2}{dx_i^2}$ where $l_i = \log(\pi(y_i|x_i))$, $i = 1, ..., \#$ data

— Markov and computational properties (on $\boldsymbol{Q}$) are preserved

— $\widetilde{\pi}(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})$ costs

- temporal: $O(n)$
- spatial: $O(n\log(n))$

If $\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta}$ is *Gaussian*, the “approximation” is exact.

INLA, $\pi(\theta_j|\boldsymbol{y})$

— Considering

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{\pi(\boldsymbol{\theta}, \boldsymbol{x}|\boldsymbol{y})}{\pi(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})}$$

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INLA, $\pi(\theta_j|\boldsymbol{y})$

— Considering

$$
\begin{aligned}
\pi(\boldsymbol{\theta}|\boldsymbol{y}) &= \frac{\pi(\boldsymbol{\theta}, \boldsymbol{x}|\boldsymbol{y})}{\pi(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})} \\
&\propto \frac{\pi(\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta})}{\pi(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y})}
\end{aligned}
$$

INLA, $\pi(\theta_j|\boldsymbol{y})$

— Considering

$$\begin{aligned}\pi(\boldsymbol{\theta}|\boldsymbol{y}) &= \frac{\pi(\boldsymbol{\theta},\boldsymbol{x}|\boldsymbol{y})}{\pi(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{y})} \\ &\propto \frac{\pi(\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{\theta})}{\pi(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{y})}\end{aligned}$$

— Gaussian approximation to denominator

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) \approx \frac{\pi(\boldsymbol{\theta})\pi(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{\theta})}{\pi_{\mathrm{G}}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{y})}\Big|_{\boldsymbol{x}=\boldsymbol{x}^*(\boldsymbol{\theta})}$$

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— mode of $\tilde{\pi}(\boldsymbol{\theta}|\boldsymbol{y})$ (optimization)
- explore $\tilde{\pi}(\boldsymbol{\theta}|\boldsymbol{y})$
  — approach $\pi(\theta_j|\boldsymbol{y})$ (numerical integration)

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INLA, $\pi(x_i|\boldsymbol{y}, \boldsymbol{\theta})$

Approaching $\pi(x_i|\boldsymbol{y}, \boldsymbol{\theta})$

- Problem
  - dim($\boldsymbol{x}$)=n is not small
  - n marginals to compute
- Laplace approximation

$$\widetilde{\pi}(x_i \mid \boldsymbol{y}, \boldsymbol{\theta}) \approx \left.\frac{\pi(\boldsymbol{x}, \boldsymbol{\theta}|\boldsymbol{y})}{\widetilde{\pi}_{GG}(\boldsymbol{x}_{-i}|x_i, \boldsymbol{y}, \boldsymbol{\theta})}\right|_{\boldsymbol{x}_{-i}=\boldsymbol{x}^*_{-i}(x_i,\boldsymbol{\theta})}$$

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INLA, $\pi(x_i|\boldsymbol{y},\boldsymbol{\theta})$

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- simpler/cruder (fast) approximation (from $\pi_G(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{\theta})$)

$$\hat{\pi}(x_i|\boldsymbol{y},\boldsymbol{\theta}) = N(x_i;\mu_i(\boldsymbol{\theta}),\sigma_i^2(\boldsymbol{\theta}))$$

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INLA, $\pi(x_i|\boldsymbol{y})$

Approaching $\pi(x_i|\boldsymbol{y},\boldsymbol{\theta})$

- integrate $\boldsymbol{\theta}$ out from $\widetilde{\pi}(x_i \mid \boldsymbol{y},\boldsymbol{\theta})$
- select values for $\boldsymbol{\theta}$
- use weighted sum

$$\widetilde{\pi}(x_i \mid \boldsymbol{y}) \propto \sum_j \widetilde{\pi}(x_i|\boldsymbol{y},\boldsymbol{\theta}_j) \times \widetilde{\pi}(\boldsymbol{\theta}_j|\boldsymbol{y})$$

Remarks

1. Expect $\widetilde{\pi}(\boldsymbol{\theta}|\boldsymbol{y})$ to be accurate, since
   - $\boldsymbol{x}|\boldsymbol{\theta}$ is *a priori* Gaussian
   - Likelihood models are 'well-behaved' so

     $$\pi(\boldsymbol{x} \mid \boldsymbol{\theta}, \boldsymbol{y})$$

     is *almost* Gaussian.
2. There are no distributional assumptions on $\boldsymbol{\theta}|\boldsymbol{y}$
3. Similar remarks are valid to

$$\widetilde{\pi}(x_i \mid \boldsymbol{\theta}, \boldsymbol{y})$$

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How can we assess the error in the approximations?

Tool 1: Compare a sequence of improved approximations

1. Gaussian approximation
2. Simplified Laplace
3. Laplace

No big differences $\rightarrow$ good approximation

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How can we assess the error in the approximations?

Tool 2: Estimate the "effective" number of parameters as defined in the Deviance Information Criteria:

$$p_{\mathrm{D}}(\theta) = \overline{D}(\boldsymbol{x};\theta) - D(\overline{\boldsymbol{x}};\theta)$$

and compare this with the number of observations
Low ratio is good.
This criteria has theoretical justification.

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