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Spacetime models in R-INLA

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Outline

Separable space-time models

Infant mortality in Paraná

PM-10 concentration in Piemonte, Italy

Multivariate dynamic regression model

— $\boldsymbol{y}_t$: n observations at time t, $E(\boldsymbol{y}_t) = \boldsymbol{\mu}_t$

$$\begin{aligned} \boldsymbol{\mu}_t &= g^{-1}(\text{diag}(\boldsymbol{F}'_t(\boldsymbol{x}_t + \boldsymbol{\mu}_x))) \\ \boldsymbol{x}_t &= \boldsymbol{G}_t \boldsymbol{x}_{t-1} + \boldsymbol{\omega}_t \end{aligned} \quad (1)$$

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- diag($\cdot$): only diagonal of $\boldsymbol{F}'_t \boldsymbol{x}_t$ counts
- $g(\cdot)$: link function, $g^{-1}(\cdot)$: inverse link
- $\boldsymbol{F}_t$: $p \times n$ covariate matrix at each time t
- $\boldsymbol{x}_t$: $p \times n$ latent (unobservable) states
- $\boldsymbol{G}_t$: $p \times p$ matrix to describe time evolution
- $\boldsymbol{\omega}_t$: $p \times n$ dimensional vector of errors
- $E(\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{\mu}_x$ are fixed effects

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Remarks

- ω_t: p vectors $\{\omega_{t1}, ..., \omega_{tp}\}$, each with length n
- each vector in $\{\omega_{t1}, ..., \omega_{tp}\}$, $\omega_{tj} \sim \text{MVNormal}(\mathbf{0}, \boldsymbol{\Sigma}_j)$
- possible in `R-INLA`
  - $\boldsymbol{y}_t$: several likelihoods
  - $\boldsymbol{\Sigma}_k$: some spatial models
  - $\boldsymbol{G}_t = \boldsymbol{G}$ (fixed over time), and diagonal: AR(1) for each state
- implementation in `R-INLA`
  - kronecker product model (for some models)
  - 'facked' zero observations (for all and 2nd dynamic models)

Kronecker product models

- $\boldsymbol{x} = \{x_{11}, ..., x_{n1}, x_{12}, ..., x_{nT}\}$
- assume

$$\pi(\boldsymbol{x}) \propto (|\boldsymbol{Q1} \otimes \boldsymbol{Q2}|^{*})^{1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}^{T}\{\boldsymbol{Q1} \otimes \boldsymbol{Q2}\}\boldsymbol{x}\right)$$

where $|.|^{*}$ is the generalized determinant

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- kronecker product model example in `R-INLA`

```
f(spatial, model='besagproper2',
  group=time, control.group=list(model='ar1'))
```

Spacetime interactions

— kronecker product models follows Clayton's rule

— combine ***Q*1** and ***Q*2** available

— warning care when main effects are in the model

— WARNING super care when ***Q*1** and/or ***Q*2** have rank deficiency

— the described dynamic model is type IV and uses ***Q*2** as AR(1)

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Infant mortality model

— infant death at municipality i and year t

$$y_{it} \sim \text{Poisson}(\text{E}_{it} \text{e}^{\eta_{it}})$$

— E_i: expected number of death (under some suposition)

- overal ratio

$$r_0 = \frac{\sum_{it} y_{it}}{\sum_{it} borns_{it}}$$

- $E_{it} = r_0 borns_{it}$
- E_{it}: expected deaths if the ratio is the same (over space and time)
- observed relative risk

$$SMR_{it} = \frac{y_{it}}{E_{it}}$$

Model structure

- linear predictor evolution over time

$$x_{it} = \rho x_{i,t-1} + s_{it}$$

- s_{it} at each time $\rightarrow$ spatially correlated

$$s_{it} | s_{-i,t} \sim N(\sum_{j \sim i} s_{j,t}/n_i, \sigma_s^2/n_i)$$

- space-time precision matrix implied: $\boldsymbol{Q} = \boldsymbol{Q_T} \otimes \boldsymbol{Q_S}$
- both smooth over time and space (if ρ is near 1)
- the full model (type IV)

$$\eta_{it} = \alpha_0 + e_t + u_i + v_t + s_i + x_{it}$$

where

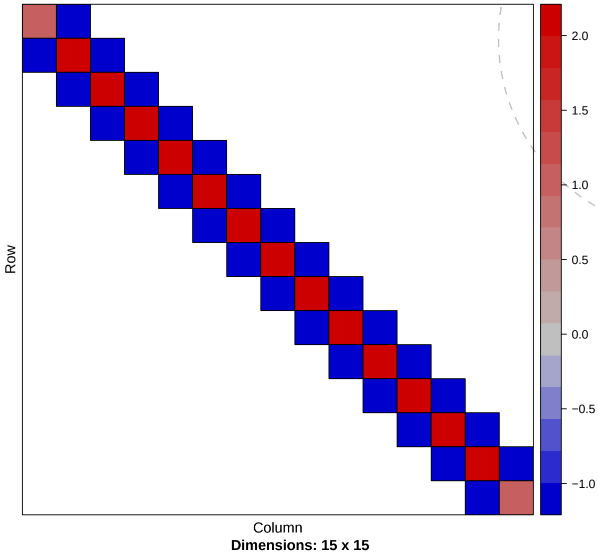
- α_0 is the intercept
- e_t is a unstructured temporal random effect
- u_i is a unstructured spatial random effect
- v_t is a structured temporal random effect
- s_i is a structured temporal random effect

On space-time random effect

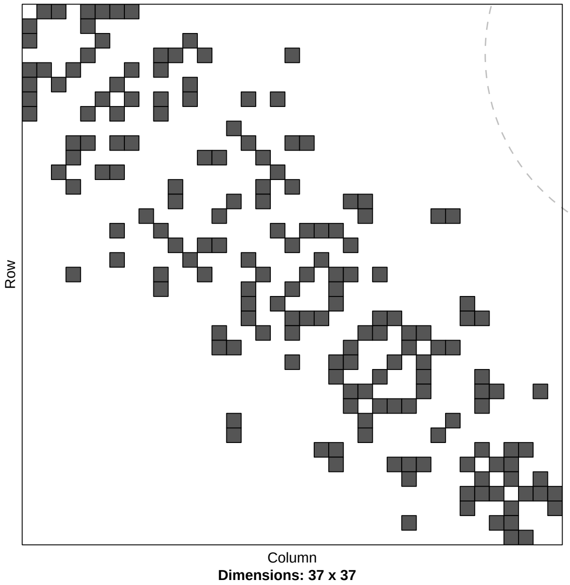
— it can be one of the four type interaction models

— dynamic model using the `besagproper2` model for space

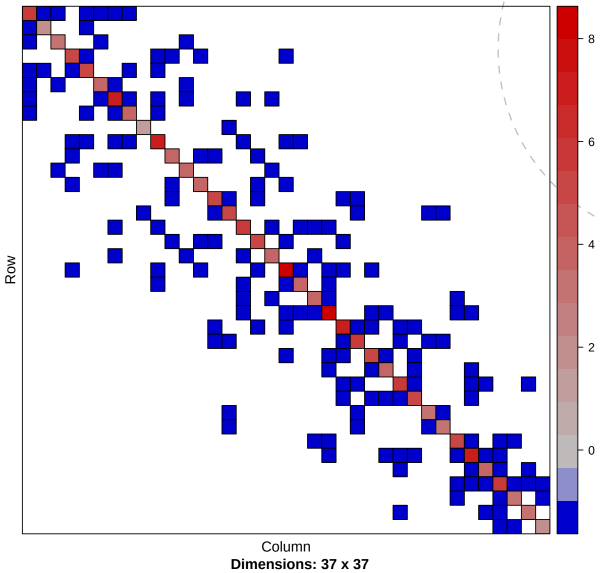
- λ = 0: no spatial structure
- λ = 1: equals the intrinsic Besag
- ρ = 0: no temporal structure
- ρ = 1: equals RW1
- $\rightarrow$ includes all the four interaction types



Temporal precision structure (for $\boldsymbol{Q}_T$)



Spatial adjacency matrix (used to build $\boldsymbol{Q}_S$)



Spatial precision structure (for $\boldsymbol{Q}_S$)

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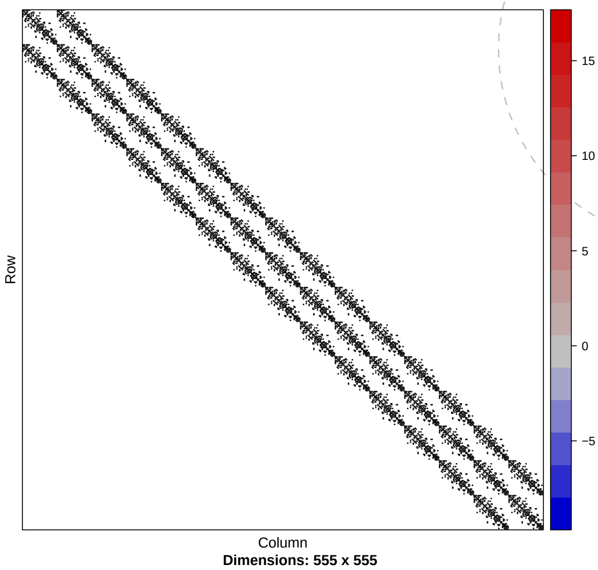

Spatio temporal precision structure (for ***Q***)

Five models for x_{it}

m_0 :	x_0	same ratio over space and time
m_1 :	$x_0 + x_{0,t}$	different ratio over time
m_2 :	$x_0 + x_{i,0}$	differet ratio over space
m_3 :	$x_0 + x_{0,t} + x_{i,0}$	common time trend + common sp. surface
m_4 :	$x_0 + x_{it}$	variation over space and time

Five models for x_{it}

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```
f0 <- y ~ 1
f1 <- y ~ 1 + f(t, model="ar1")
f2 <- y ~ 1 + f(i, model="besag", graph="map/cwbm.graph")
f3 <- y ~ 1 + f(t, model="ar1") +
    f(i, model="besag", graph="map/cwbm.graph")
f4 <- y ~ 1 + f(i, model="besag", graph="map/cwbm.graph",
           group=t, control.group=list(model="ar1"))
```

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Space-time dynamic intercept

— The (linear) measurement equation

$$\boldsymbol{y}_{it} = F_{it}'\beta + \boldsymbol{A}_{i(t)}\boldsymbol{x}_t + \epsilon_{it}$$

- $\boldsymbol{F}_t$ is a matrix of covariates
- β are the fixed effects
- $\boldsymbol{A}_{(t)}$ picks out the appropriate values of $\boldsymbol{x}_t$
- $\epsilon_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2 \boldsymbol{I})$

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- vector AR(1) process for $\boldsymbol{x}$

$$\boldsymbol{x}_t = \rho\boldsymbol{x}_{t-1} + \boldsymbol{\omega}_t$$

  - $\boldsymbol{\omega_t}$: spatial SPDE model

$$\boldsymbol{\omega}_t \overset{\text{i.i.d.}}{\sim} N(\boldsymbol{0}, \boldsymbol{Q}^{-1}),$$

  - ρ is the time correlation

PM-10 concentration in Piemonte, Italy

Cameletti *et al.* (2011), on `r-inla.org`

— 24 monitoring stations

— Daily data from 10/05 to 03/06

Space model part

— Make the mesh

```
mesh <- inla.mesh.2d(points =NULL,
                     points.domain=borders,
                     offset=c(10, 140),
                     max.edge=c(40,1000))
```

— Make the latent model

```
spde = inla.create.spde(mesh,model="matern")
```

Using the `group` feature

— Construct a kronecker product model using the `group` feature

```
formula = y ~ -1 + intercept + WS + HMIX + ... +
   f(field, model=spde,
      group =time,
      control.group=list(model="ar1")
     )
```

— This tells INLA that the observations are grouped in a certain way.

— `control.group` contains the grouping model (`ar1`, `exchangable`, `rw1`, and others) as well as their prior specifications.

Make an *A* matrix

— Use the `group` argument

```
LocationMatrix = inla.spde.make.A(mesh = mesh,
        loc =dataLoc, group=time, n.group=nT)
```

— data locations in all `group=time` level

— builds an *A* matrix in an appropriate way

Organising the data

Covariates at the data points, but the latent field only defined their through the A matrix

We need to make sure that ***A*** *only applies to the random effect.*

```
idx.set <- inla.spde.make.index("mesh.idx",n.field=nmesh,
                  n.group=T)
stack = inla.stack( data = dat,
       A = list(1, LocationMatrix),
       effects = list( list(WS = cov$WS,...),
            c(idx.set,
                       list(intercept=rep(1,mesh$n*nT)))
       )
    )
```