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# A random matching theory ☆

C.D. Aliprantis [a], G. Camera [a,*], D. Puzzello [b]

[a] *Department of Economics, Purdue University, West Lafayette, IN 47907-2056, USA*
[b] *Departments of Economics and Mathematics, University of Kentucky, Lexington, KY 40506-0034, USA*

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**Abstract**

We develop theoretical underpinnings of pairwise random matching processes. We formalize the mechanics of matching, and study the links between properties of the different processes and trade frictions. A particular emphasis is placed on providing a mapping between matching technologies and informational constraints.


*JEL classification:* C00; C78; D83; E00

*Keywords:* Random matching; Frictions; Spatial interactions

## 1. Introduction

A large segment of the economic literature is concerned with the study of allocations that arise when markets are not well-functioning. A defining characteristic of this literature is its focus on informational and spatial frictions, and the desire to make them explicit by assuming that economic interactions occur in small coalitions. To this end, the literature has traditionally relied on pairwise random matching frameworks. This basic modeling tool has found use in a wide variety of settings, from the study of social norms (as in Kandori, 1992), to unemployment (as in Mortensen and Pissarides, 1994), to business cycles (as in Diamond and Fudenberg, 1989), and

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* Corresponding author.
*E-mail addresses:* aliprantis@mgmt.purdue.edu (C.D. Aliprantis), gcamera@mgmt.purdue.edu (G. Camera), daniela.puzzello@uky.edu (D. Puzzello).

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to the foundations of monetary theory (as in Kiyotaki and Wright, 1989; Shi, 1997; and Green and Zhou, 2002).

A limitation of this literature is that the treatment of matching—as a technology—is mostly descriptive and insufficiently formalized. For example, the mechanics of the economic interactions are generally not made explicit or the map between matching and the frictions assumed to be in place is open to various interpretations. This tends to prevent a clear understanding of how the matching technology impairs market functioning, and consequently the possible allocations. These limitations must be overcome to better formulate models of economies with frictions. An objective economic analysis is thought of as one that focuses on the allocations predicted using a carefully specified physical environment (preferences, technologies, etc.). Thus, a comprehensive theory of exchange cannot be derived by simply assuming that certain economic interactions may or may not take place. Ideally, the theory should clarify how the trading or institutional constraints assumed to be in place originate in the underlying economic environment.

The purpose of this study is to build a more solid foundation for random matching models, by means of a set-theoretic approach. There are two major contributions. First, the paper provides a formalization of the *mechanics* of random pairwise matching. To do so, it uses as a starting point the approach to deterministic matching provided by Aliprantis et al. (2006). Compared to that paper, this study introduces the concept of spatial separation in terms of population partitions, and uses probability measures to match agents only within the same partition set. This provides a clear and simple formalization of random matching. By focusing on the technological aspects of meeting processes, this study adds to a literature on matching models.[1]

A second contribution of this investigation is that it spells out how different matching technologies may facilitate (or obstacle) the exchange of economic resources and information among agents. Particular emphasis is paid to formalizing how the matching technology's properties affect the level of informational isolation that exists in economies where agents are randomly paired over time.[2] Indeed, this is what especially differentiates our work from previous studies on random matching processes; see, for instance, the matching scheme of Boylan (1992) for a countable population.

The technical procedure that we use to construct any random matching process involves three basic steps. The first step is to specify how to divide the population in each period into spatially separated clusters of agents. To do so, we use partitional correspondences. Then, one must define and calculate all possible ways to form pairs in each cluster. In this case, we resort to using a class of permutation functions, the so-called involutions. Finally, for each period one must specify a probability measure over all possible pairings, for each cluster. This gives us the desired random matching rule for a cluster, and a well-defined random matching process for the entire population in each period. A pairwise random matching framework can then be formalized as a sequence of partitional correspondences, involutions and probability measures. Given these sequences, we can then explicitly specify matching histories, and therefore we can formalize the degree of informational isolation that exists among agents.

The paper is organized as follows. Section 2 introduces the mathematical background. Sections 3 and 4 discuss pairwise random matching in a single period and over time and characterize

[1] For example, see Ioannides (1990), Gilboa and Matsui (1992), and the more recent works of Duffie and Sun (in press, 2004) on the exact law of large numbers for random pairwise matching.

[2] Research that has taken into consideration these concerns has appeared especially in the monetary literature. For instance, see the works of Huggett and Krasa (1996), Kocherlakota (1998), and Corbae and Ritter (2004).

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matching mechanisms according to the degree of informational isolation they can sustain. Section 5 demonstrates how random matching economies can be constructed in which traders are completely anonymous. Section 6 presents an application of our theoretical construct to random matching models of money. Section 7 concludes.

## 2. Mathematical background

If $A$ is an arbitrary set, then $|A|$ denotes its cardinality. As usual, $|A| = \mathcal{N}_0$ means that $A$ is countable and $|A| = \mathfrak{c}$ indicates that the cardinality of $A$ is the continuum. If $A$ is a union of a pairwise disjoint family of sets $\{A_i\}_{i\in I}$, then we denote it by $A = \bigsqcup_{i\in I} A_i$. If $A = \bigsqcup_{i\in I} A_i$, then we say that the family $\{A_i\}_{i\in I}$ *partitions* the set $A$.

**Definition 1.** A *correspondence* $\psi$ from a set $X$ to a set $Y$ assigns to each $x$ in $X$ a subset $\psi(x)$ of $Y$. We write $\psi : X \twoheadrightarrow Y$ to distinguish a correspondence from a function.

We use the correspondence concept since we intend to divide a population $X$ into separate clusters of agents. To do so, we focus on correspondences with $X = Y$, that is $\psi : X \twoheadrightarrow X$. Furthermore, to formalize the notion of spatial separation of clusters, we consider a special class of correspondences.

**Definition 2.** A correspondence $\psi: X \twoheadrightarrow X$ is *partitional* if (a) $x \in \psi(x)$ for every $x \in X$, and (b) whenever $y \in \psi(x)$, then $\psi(y) = \psi(x)$. If, in addition, $|\psi(x)| = k$ for all $x \in X$, then we say that $\psi$ is *k-partitional.*

This definition mirrors the one in Osborne and Rubinstein (1999, p. 68). It states that an agent $x$ always belongs to the cluster $\psi(x)$ and any two clusters either coincide or are disjoint.[3] One can interpret this as meaning that there is spatial separation among clusters. To see this note that, by (b), if some agent $y$ belongs to $\psi(x)$, then $x$ and $y$ must be in the same cluster.

We use a correspondence $\psi$ to partition the population into subsets of spatially separated groups of agents, called clusters. Whether and how agents in a cluster can interact with each other, depends on the matching rule. To formalize this, we use permutations. A *permutation* of a non-empty set $X$ is a one-to-one function $\phi$ from $X$ onto $X$. If $X$ is a finite set, say $X = \{x_1, x_2, \ldots, x_k\}$, then a permutation $\phi$ on $X$ is a matrix

$$\phi = \begin{pmatrix} x_1 & x_2 & \ldots & x_k \\ y_1 & y_2 & \ldots & y_k \end{pmatrix},$$

where $y_j = \phi(x_j) \in X$ and $y_j \neq y_i$ if $i \neq j$. If $\phi$ is such that $\phi^2 = \phi \circ \phi = I$ on $X$, that is, if the function $\phi$ composed with itself is the identity function, then $\phi$ is called an *involution*; see Shashkin (1991). Involutions formalize the concept of bilateral matchings.

## 3. Random matching in a period

In this section, we discuss how to match agents randomly in any representative period. Thus, we omit the time subscript. We adopt a procedure that involves three separate steps. First, using

[3] If $z \in \psi(x) \cap \psi(y)$, then $\psi(z) = \psi(x)$ since $z \in \psi(x)$ and $\psi(z) = \psi(y)$ since $z \in \psi(y)$. Thus, $\psi(x) = \psi(y)$.

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partitional correspondences, we specify how to divide the population into spatially separated clusters of agents. Then, using involutions, we define and calculate all possible ways to form pairs in each cluster. Finally, for each cluster, we specify a probability measure over all possible pairings. This gives us the desired random matching rule for a cluster and a random matching process for the population.

### *3.1. Step 1. Spatial separation using clustering rules*

Since we want to deal with matches that are separated in space, we start by formalizing the notion of spatial separation. To achieve this, we introduce the concept of a $k$-clustering rule to divide the population $X$ into clusters of $k$ individuals each. Later, we will formalize a notion of spatial separation for these clusters.

**Definition 3.** A *$k$-clustering rule* for a population $X$ is a $k$-partitional correspondence $\psi: X \twoheadrightarrow X$. We call $\psi(x)$ the *cluster* of $x$.

Every $k$-clustering rule $\psi$ induces a partition on the population $X$ by selecting $k$ agents at a time that are placed in separate groups. That is, $\psi$ partitions $X$ into 'slices' or equivalence classes.[4] The family of equivalence classes is denoted $\{X_s\}_{s\in S}$, where $S$ is the index set of all slices. In other words, for each $s \in S$ there exists some $x \in X$ such that $\psi(x) = X_s$. For instance, if $X = \{a, b, c, d\}$ and $\psi$ generates the clusters $\psi(a) = \psi(b) = \{a, b\}$ and $\psi(c) = \{c, d\}$, then $S = \{1, 2\}$ where $X_1 = \{a, b\}$ and $X_2 = \{c, d\}$.

The natural question at this point is whether we can construct clustering rules on any set. It turns out that not all populations can be partitioned according to a $k$-clustering rule. The next result establishes a basic condition under which this can be done: it requires to partition the population $X$ into $k$ subsets of identical cardinality.

**Theorem 4** *(Existence of clustering rules). If $A_1, A_2, \dots, A_k$ are pairwise disjoint sets having the same cardinality and $X = \bigsqcup_{i=1}^{k} A_i$, then there exists a $k$-partitional correspondence $\psi: X \twoheadrightarrow X$. In particular, $\psi$ can be chosen so that for each $x \in X$ the set $\psi(x)$ consists of $k$-elements one from each set $A_i$.*

**Proof.** Since the sets $A_i$ have the same cardinality, for each $i = 1, \dots, k-1$, we can find a function $f_i: A_i \to A_{i+1}$ which is one-to-one and surjective (onto). We claim the following: If $2 \leqslant j \leqslant k$ and $x \in A_j$, then there exists a unique element $r_x \in A_1$ (called the *root* of $x$) such that $x = f_{j-1} f_{j-2} \cdots f_1(r_x)$. Indeed, note that the element $r_x = (f_1^{-1} \cdots f_{j-1}^{-1})(x)$ satisfies the desired property, that is,

$$\text{If } x = f_{j-1} f_{j-2} \cdots f_1(x_1), \quad \text{where } x_1 \in A_1 \text{ and } 2 \leqslant j \leqslant k, \quad \text{then } r_x = x_1. \tag{$\star$}$$

The uniqueness of $r_x$ should be obvious. If $x \in A_1$, then we let $r_x = x$.

[4] We note that given any partition, there is exactly one equivalence relation on $X$ from which it is derived. An *equivalence relation* on a set $X$ is a binary relation $\sim$ on $X$ satisfying the following three properties: (1) (reflexivity) $x \sim x$ for every $x \in X$; (2) (symmetry) If $x \sim y$, then $y \sim x$; and (3) (transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$. Given an equivalence relation $\sim$ on a set $X$ and an element $x \in X$, the *equivalence class* of $x$ is the subset of $X$ defined by $[x] = \{y \in X\colon\ y \sim x\}$. Note that $x \in [x]$ for all $x \in X$, and any two equivalence classes are either disjoint or equal. Given an equivalence relation on $X$, the collection of all equivalence classes determined by $\sim$ is a partition of $X$. Thus, studying equivalence relations is equivalent to studying partitions.

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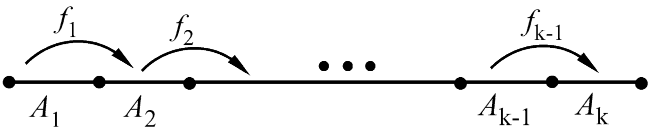


Fig. 1. The $k$-clustering rule of Theorem 4.

Next, define $\psi : X \twoheadrightarrow X$ by $\psi(x) = \{r_x, f_1(r_x), f_2 f_1(r_x), \ldots, f_{k-1} f_{k-2} \cdots f_1(r_x)\}$. It should be clear that $\psi(x)$ contains $k$ elements such that $\psi(x) \cap A_j$ is a singleton for each $j = 1, 2, \ldots, k$. That is, $\psi(x)$ consists of all elements of $X$ that have $r_x$ as their root. (Clearly, $x \in \psi(x)$ and $\psi(x)$ consists exactly of one element from each $A_i$.) To prove that $\psi$ is a $k$-partitional correspondence, it remains to be shown that if $y \in \psi(x)$, then $\psi(y) = \psi(x)$. There are two cases to consider:

(1) $y = r_x \in A_1$. In this case, we have $r_y = r_x$, and so

$$\begin{aligned} \psi(y) &= \big\{r_y, f_1(r_y), \ldots, f_{k-1} f_{k-2} \cdots f_1(r_y)\big\} \\ &= \big\{r_x, f_1(r_x), \ldots, f_{k-1} f_{k-2} \cdots f_1(r_x)\big\} = \psi(x). \end{aligned}$$

(2) $y \neq r_x$. Here we have $y = f_{j-1} f_{j-2} \cdots f_1(r_x)$ for some $2 \leqslant j \leqslant k$. This, in conjunction with ($\star$), yields $r_y = r_x$. Thus, as above, $\psi(y) = \psi(x)$.

This completes the proof. □

An illustration of the clustering rule described in Theorem 4 is shown in Fig. 1.

We note that $k$-clustering rules, if they exist, are not necessarily unique. This is due to the flexibility in the selection of agents from each of the $k$ sets which define the partition. Clearly, we have many choices over the functions $f_i$, $1 \leqslant i \leqslant k-1$, as long as they are one-to-one and onto. A different $k$-partitional correspondence is generated by a different choice of any of the $f_i$.

What if the population $X$ cannot admit $k$-clustering rules? Then, we can 'normalize' $X$ (as long as it has at least $k$ agents) so that a $k$-clustering rule can be constructed on a *subset* of $X$. The remaining agents are assigned to clusters of one agent each.

**Corollary 5.** *Let $X = (\bigsqcup_{i=1}^{k} A_i) \sqcup A_0 = Y \sqcup A_0$, where $A_0, A_1, \ldots, A_k$ are nonempty pairwise disjoint sets and $A_1, \ldots, A_k$ have the same cardinality. Then we can construct a partitional correspondence $\psi : X \twoheadrightarrow X$ such that* (i) *$\psi$ on $Y$ is $k$-partitional, and* (ii) *$\psi$ on $A_0$ is* 1*-partitional.*

Notice that we can always partition an infinite population $X$ as in Corollary 5. Now that we know how to group an arbitrary population $X$ into clusters of agents, we study how to pair agents in each cluster. In this way, we can also formalize a notion of spatial separation for any economy.

### *3.2. Step 2. Bilateral matching using involutions*

Suppose we have divided the population $X$ according to $\psi$ into the clusters $\{X_s\}_{s \in S}$. We want to pair agents only within each cluster $X_s$. To do so, we use involutions to define bilateral matching rules on any set $\Omega$.

**Definition 6.** A *bilateral matching rule* for a set of agents $\Omega$ is an involution of $\Omega$.

Recall that a permutation of $\Omega$ is any one-to-one and onto function $\phi$ on $\Omega$. This means that any permutation can assign an agent to himself. However, such a permutation need not be consistent with the idea of bilateral matching. For example, if $\Omega = \{a, b, c\}$, then a permutation may assign $a$ to $b$, $b$ to $c$, and $c$ to $a$, which clearly is not a matching. Therefore, we need the "involution" restriction: the inverse of the permutation $\phi$ must coincide with itself or $\phi^2 = I$.

Now that we know what is a bilateral matching rule, we have a natural way to formalize the notion of spatial separation in the economy.

**Definition 7.** A *spatially separated economy* is a triplet $(X, \psi, \phi)$ such that:

(a) $\psi : X \twoheadrightarrow X$ is a $k$-clustering rule on $X$, and
(b) $\phi : X \to X$ is a bilateral matching rule that leaves each cluster $X_s$ invariant, that is, $\phi(X_s) \subseteq X_s$ for each $s \in S$.

Clusters of agents are spatially separated if an agent $y$ belonging to a cluster $\psi(x)$ can only meet an agent who also belongs to $\psi(x)$. A bilateral matching rule does not necessarily match every agent to someone else, in his own cluster. We say that a bilateral matching rule on $\Omega$ is *exhaustive* if no agent in $\Omega$ is unmatched, that is if $\phi(\omega) \neq \omega$ for all $\omega \in \Omega$. Of course, several bilateral matching rules exist, exhaustive or not. Thus, it is natural to ask how many possible pairings of the $k$ agents in $\Omega$ can be accomplished.

**Definition 8.** The collection of all bilateral matching rules on $\Omega$ is denoted $\mathcal{B}(\Omega)$.

For the rest of this paper, $\Omega = \{\omega_1, \ldots, \omega_k\}$ will denote a finite set of $k$ agents. Notice that $\mathcal{B}(\{\omega_1, \ldots, \omega_k\})$ consists of all possible ways in which the $k$ agents in $\Omega$ can be bilaterally arranged—either by pairing them with someone else or with themselves. Since the cardinality of the set of all possible permutations of $\Omega$ is $k!$ and $\mathcal{B}(\Omega)$ is a subset of the set of all permutations, it follows that $\mathcal{B}(\Omega)$ is also a finite set whose cardinality is less than $k!$. The number of possible bilateral matching rules in $\Omega$ can be determined recursively as follows.

**Lemma 9.** *If $\ell_k = |\mathcal{B}(\Omega)|$ is the number of all possible bilateral matching rules on a set $\Omega$ with $k$ agents, then $\ell_1 = 1$, $\ell_2 = 2$, and $\ell_{k+1} = \ell_k + k\ell_{k-1}$ for $k \geqslant 2$.*

**Proof.** It is obvious that there is only one way to arrange one agent and two ways to arrange two agents. Thus $\ell_1 = 1$ and $\ell_2 = 2$. Now, suppose we have $k + 1 \geqslant 3$ agents in a cluster. There are two possibilities: (1) agent $k + 1$ is matched to himself, and (2) agent $k + 1$ is matched to someone else. In case (1), according to the definition of possible bilateral matching rules, the remaining $k$ agents can be matched in $\ell_k$ different ways. In case (2) agent $k + 1$ can be matched to any one of the other $k$ agents. The remaining $k - 1$ agents can be matched in $\ell_{k-1}$ different possible ways. Therefore $\ell_{k+1} = \ell_k + k\ell_{k-1}$. □

We can also calculate the number of possible exhaustive bilateral matching rules for a cluster of $k$ agents.

**Lemma 10.** *Let $n_k$ and $e_k$ denote the number of possible non-exhaustive and exhaustive bilateral matching rules on a set $\Omega$ of $k \geqslant 2$ agents. Then*:

(a) *$n_1 = 1$, $n_2 = 1$ and $n_{k+1} = \ell_k + kn_{k-1}$ for $k \geqslant 2$.*
(b) *$e_1 = 0$, $e_2 = 1$ and $e_{k+1} = ke_{k-1}$ for $k \geqslant 2$.*

*Furthermore, $e_{2k+1} = 0$ and $e_{2k} = \frac{(2k)!}{2^k k!}$ for $k = 1, 2, 3, \ldots$.*

**Proof.** It is obvious that $n_1 = n_2 = 1$ since in both cases only the identity is non-exhaustive. If $k \geqslant 2$, then there are two possibilities: agent $k + 1$ is matched to himself or to someone else. In the first case the remaining $k$ agents can be paired in $\ell_k$ different ways. Otherwise, agent $k + 1$ can be paired to any of the $k$ agents and the remaining $k - 1$ agents can be paired in $n_{k-1}$ different possible non-exhaustive ways. Therefore $n_{k+1} = \ell_k + kn_{k-1}$.

Now notice that the number of possible exhaustive pairings is just the difference between the total number of possible pairings and the number of all possible non-exhaustive pairings. Thus, we have $e_1 = \ell_1 - n_1 = 0$ and $e_2 = \ell_2 - n_2 = 1$. Also,

$$e_{k+1} = \ell_{k+1} - n_{k+1} = k(\ell_{k-1} - n_{k-1}) = ke_{k-1}.$$

The latter, in conjunction with $e_1 = 0$, implies that $e_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$. By induction, it is easy to see that $e_{2k} = \frac{(2k)!}{2^k k!}$. If $k = 1$, then clearly $e_2 = \frac{2!}{2\times 1!} = 1$. For the induction step, assume that $e_{2k} = \frac{(2k)!}{2^k k!}$ is true for $k \geqslant 1$, then we have to show it is true for $k + 1$. To see this, note that using the recursive formula for $e_{k+1}$ we have $e_{2(k+1)} = (2k + 1)e_{2k} = (2k + 1)\frac{(2k)!}{2^k k!} = \frac{(2k+2)!}{2^{k+1}(k+1)!}$. $\square$

As an example, let $k = 3$ and $\Omega = \{a, b, c\}$. Then the number of all possible pairings (i.e., bilateral matching rules) is $\ell_3 = n_3 = 4$. The set consisting of all bilateral matching rules is $\mathcal{B}(\{a, b, c\}) = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where

$$\phi_1 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix},$$
$$\phi_3 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \qquad \phi_4 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}.$$

That is, there are four possible ways to pair the agents $a$, $b$ and $c$. We can leave them unmatched, which is the permutation $\phi_4$ or we can form pairs leaving one agent unmatched according to $\phi_1$, $\phi_2$ and $\phi_3$. What is interesting is that (as the next table demonstrates) even for relatively small clusters of agents the number of possible pairings is very large.

| $k$ | $\ell_k$ | $n_k$ | $e_k$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | 2 | 1 | 1 |
| 3 | 4 | 4 | 0 |
| 4 | 10 | 7 | 3 |
| 5 | 26 | 26 | 0 |
| 6 | 76 | 61 | 15 |
| 7 | 232 | 232 | 0 |
| 8 | 764 | 659 | 105 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 20 | 23,758,664,096 | 23,103,935,021 | 654,729,075 |

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An immediate consequence is that we can generate a very large number of possible pairings despite the use of finite clusters of agents. This is very convenient, since it allows us to construct bilateral matches that are random by selecting randomly one out of many possible pairings in each cluster. The mechanics of this are described in the sequel.

### 3.3. Step 3. Random pairings using probability measures

We start by formalizing a notion of a random matching.

**Definition 11.** A *stochastic bilateral matching rule* on $\Omega = \{\omega_1, \dots, \omega_k\}$ (or a *stochastic rule*) is simply a probability measure $f$ on $\mathcal{B}(\Omega)$.

Using this formalization we now show how to construct random pairings on $\Omega$.

**Lemma 12.** *Every stochastic rule $f$ on a set $\Omega = \{\omega_1, \dots, \omega_k\}$ induces a probability measure $F : \Omega \times \Omega \to [0, 1]$ via the formula*

$$F(\omega_i, \omega_h) = \sum_{\{\phi \in \mathcal{B}(\Omega):\ \omega_i = \phi(\omega_h)\}} f(\phi) = f\bigl(\{\phi \in \mathcal{B}(\Omega):\ \omega_i = \phi(\omega_h)\}\bigr).$$

*The measure $F$ satisfies the following properties*:

(i) *For all $i$ and $h$ we have $F(\omega_i, \omega_h) = F(\omega_h, \omega_i)$.*
(ii) *For each fixed $\omega_h \in \Omega$ we have $\sum_{i=1}^k F(\omega_i, \omega_h) = 1$.*
(iii) *If $k$ is odd, then $F(\omega_i, \omega_i) > 0$ for some $i$.*

*Moreover, $F$ defines a doubly stochastic matrix*[5]

| | $\omega_1$ | $\omega_2$ | $\dots$ | $\omega_k$ |
|---|---|---|---|---|
| $\omega_1$ | $F(\omega_1, \omega_1)$ | $F(\omega_1, \omega_2)$ | $\dots$ | $F(\omega_1, \omega_k)$ |
| $\omega_2$ | $F(\omega_2, \omega_1)$ | $F(\omega_2, \omega_2)$ | $\dots$ | $F(\omega_2, \omega_k)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $\omega_k$ | $F(\omega_k, \omega_1)$ | $F(\omega_k, \omega_2)$ | $\dots$ | $F(\omega_k, \omega_k)$ |

**Proof.** Part (i) follows from the fact that $\omega_i = \phi(\omega_h)$ if and only if $\omega_h = \phi(\omega_i)$. To prove part (ii), fix $\omega_h \in \Omega$ and note that $\bigsqcup_{i=1}^k \{\phi \in \mathcal{B}(\Omega):\ \omega_i = \phi(\omega_h)\} = \mathcal{B}(\Omega)$. This implies

$$\sum_{i=1}^k F(\omega_i, \omega_h) = \sum_{i=1}^k f\bigl(\{\phi \in \mathcal{B}(\Omega):\ \omega_i = \phi(\omega_h)\}\bigr) = f\bigl(\mathcal{B}(\Omega)\bigr) = 1.$$

To see (iii), observe that if $k$ is odd, then for all $\phi \in \mathcal{B}(\Omega)$ there exists some $\omega_i \in \Omega$ such that $\omega_i = \phi(\omega_i)$ and so $\mathcal{B}(\Omega) = \bigcup_{i=1}^k \{\phi \in \mathcal{B}(\Omega):\ \omega_i = \phi(\omega_i)\}$. This implies that $F(\omega_i, \omega_i) > 0$ holds true for some $i$. $\square$

A stochastic rule on $\Omega$ selects with probability $f(\phi)$ the pairings specified by the matching rule $\phi \in \mathcal{B}(\Omega)$. Since each $\phi$ assigns every agent $\omega_i \in \Omega$ to someone in $\Omega$, then we can calculate

[5] A non-negative real matrix is *doubly stochastic* if each row and column sums up to one.

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the probability that $\omega_i$ meets $\omega_h$. To do so, note that each $\phi$ in $\mathcal{B}(\Omega)$ can be considered as an independent outcome. We define the probability of a match between $\omega_i$ and $\omega_h$ as $F(\omega_i, \omega_h)$ and compute it by adding the probabilities $f(\phi)$ associated to those outcomes in which $\omega_i$ meets $\omega_h$. Looking across all possible pairings, this gives rise to the doubly stochastic matrix exhibited in the statement of Lemma 12. Clearly, from such a matrix we can always reconstruct the probability measure $f$.

Now that we know how to construct random pairings on any finite set of agents $\Omega$, we can formalize a notion of random matching for the entire population.

**Definition 13.** A *stochastic bilateral matching process* over a population $X$ relative to a $k$-clustering rule $\psi : X \twoheadrightarrow X$ is a family $\mathcal{F} = \{f_s\}_{s\in S}$ of probability measures, where $f_s$ is a stochastic rule over $\mathcal{B}(X_s)$ and $\{X_s\}_{s\in S}$ is the collection of clusters induced by $\psi$.

Briefly, here is how we randomly pair agents. We start by using a clustering rule $\psi$ to partition the population $X$ into spatially separated clusters $X_s$ of $k$ agents each.[6] Once this is done, we find all possible ways to pair agents within each cluster $X_s$, which gives rise to the set $\mathcal{B}(X_s)$. Given this, we specify a probability measure over $\mathcal{B}(X_s)$, which is our stochastic rule. The collection of all such rules for $\{\mathcal{B}(X_s)\}_{s\in S}$ is the stochastic bilateral matching process $\mathcal{F}$.[7] Thus, $\mathcal{F}$ induces a family of probability measures $\{F_s\}_{s\in S}$ satisfying the properties in Lemma 12. A single realization of this stochastic process generates a unique match of the population. We call a stochastic bilateral matching process $\mathcal{F}$ over the population $X$ exhaustive, if $F_s(\omega, \omega) = 0$ for all agents $\omega \in X_s$ and all $s \in S$. Clearly, this cannot occur if the $k$-partitional correspondence $\psi$ has $k$ odd; see Lemma 12(iii). Next we show how to construct mechanisms that pair agents randomly over time.

## 4. Random matching over time

Consider discrete time $t = 0, 1, 2, \ldots$. We start by having agents unmatched. We say that a sequence $\Psi = \{\psi_t\}_{t=0}^{\infty}$ is a *k-clustering mechanism* if $\psi_t$ is a $k$-clustering rule for the population $X$ in every period $t \geqslant 1$. In this way, we can construct random bilateral matches over time by specifying a sequence of bilateral stochastic matching processes.

**Definition 14.** A *bilateral stochastic matching mechanism* (or a *stochastic mechanism*) over a population $X$ is a quadruplet $(X, \Psi, \Phi, \mathcal{F})$, where:

(i) $\Psi$ is a $k$-clustering mechanism over $X$,
(ii) $\Phi = \{\phi_t\}_{t=0}^{\infty}$ is a sequence of bilateral matching rules on $X$ such that the triplet $(X, \psi_t, \phi_t)$ is a spatially separated economy for each $t$, and
(iii) $\mathcal{F} = \{\mathcal{F}_t\}_{t=0}^{\infty}$ is a sequence of stochastic bilateral matching processes such that:
  (a) $\mathcal{F}_0$ satisfies $\phi_0(x) = x$ for each $x \in X$, and
  (b) $\mathcal{F}_t$ is a stochastic bilateral matching process for $\psi_t$ for each $t \geqslant 1$.

[6] The index set $S$ can be countable or uncountable. For example, if $X = [0, 1]$ then there are infinitely many clusters of $k$ agents, and $S$ is uncountable. In fact, since a countable union of countable sets is countable it must be the case that $|S| = \mathfrak{c}$.

[7] Since the clusters $X_s$ are finite sets and we randomly pair agents only within the same cluster (spatial separation), questions regarding measurability issues are irrelevant here.

The stochastic mechanism gives the probability that on each date $t \neq 0$ an agent $x$ gets matched to someone in his own cluster $\psi_t(x)$. Clearly, every agent meets some other agent in each period $t \neq 0$ if $\mathcal{F}_t$ is exhaustive in each $t \neq 0$. The collection of all deterministic bilateral matching mechanisms is a subset of all stochastic mechanisms, where $\mathcal{F}_t$ induces a family of degenerate probability measures in each period.

It is convenient to call the agents in $\psi_t(a)$ the *clustermates* of $a$ in period $t$. Among these agents, there is only one agent $\phi_t(a) \in \psi_t(a)$ who is the *partner* of $a$ in period $t$. To account for the information that may be available in a match, we need to examine the agents' matching histories. To do so, one must keep track of the clusters to which paired agents belong on each date. We denote by $P_t(a)$ the set of all clustermates of $a$ (including $a$ himself) in periods up to and including $t$. That is,

$$P_t(a) = \bigcup_{\tau=0}^{t} \psi_\tau(a).$$

While $P_t(a)$ accounts for all agents that belonged to the same clusters to which $a$ belonged, it excludes agents that were clustermates of $a$'s clustermates and partners, and so on. There is an easy way to keep track of all these 'indirect' connections among agents in a recursive manner. Denote by $\Pi_t(a)$ the set of $a$'s past and current clusters, the clusters to which $a$'s current clustermates belonged in the past, and so on. That is, let

$$\Pi_t(a) = \begin{cases} P_0(a) & \text{for } t = 0, \\ \bigcup_{b \in \psi_t(a)} \Pi_{t-1}(b) & \text{for } t \geqslant 1. \end{cases}$$

By an inductive argument, we can see that $P_t(a) \subseteq \Pi_t(a)$ and, although $\Pi_t(a)$ is a very large set, it is finite since it is a finite union of finite sets. Also, $\Pi_t(a)$ does not include anyone who has been in a cluster with $a$'s partners (or clustermates) after these agents moved away from agent $a$.

Why do we need all this complex machinery? The reason is now that we know how to match agents over time, we want to be able to discuss how the matching technology in place affects the flow of information in the marketplace. That is, we want to make explicit how different matching mechanisms generate (or remove) obstacles to informational flows.

This issue deals with the broadly defined notion of 'anonymity' in trade, which is often seen as a central assumption in several models of matching.[8] The question we need to answer at this point is the following: what does it exactly mean for matched agents to be anonymous? To formalize a notion of anonymity, we need to take two steps. First, we must know how to look into an agent's past. This was already done by introducing the sets $P_t(a)$ and $\Pi_t(a)$, which trace the matching history of everybody, as in Kocherlakota (1998). Second, we need to formalize how these matching histories can be used to define the information that can be available to agent in a match. This will be done next.

**Definition 15.** A $k$-clustering mechanism $\Psi$ on the population $X$ is said to be:

(1) *Eventually weakly anonymous*, if for each $a \in X$ there is some $t \geqslant 0$ such that
  (i) $\psi_{\tau'}(a) \cap \psi_\tau(a) = \{a\}$ for all $\tau', \tau \geqslant t$ with $\tau' \neq \tau$, and
  (ii) $P_t(a) \cap [\bigcup_{\tau=t+1}^{\infty} \psi_\tau(a)] = \{a\}$.

[8] For instance, anonymity is a prominent feature in the foundations of the money literature (e.g., see Ostroy, 1973) and the social norms literature (e.g., see Kandori, 1992).

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(2) *Weakly anonymous*, if for all $a \in X$, all $t \geqslant 0$ and all $\tau \neq t$ we have

$$\psi_t(a) \cap \psi_\tau(a) = \{a\}.$$

(3) *Anonymous*, if for all $a \in X$, all $t \geqslant 0$ and all $b \in \psi_{t+1}(a)$ with $b \neq a$ we have

$$P_t(a) \cap P_t(b) = \emptyset.$$

(4) *Strongly anonymous*, if for all $a \in X$, all $t \geqslant 0$ and all $b \in \psi_{t+1}(a)$ with $b \neq a$ we have

$$\Pi_t(a) \cap \Pi_t(b) = \emptyset.$$

We say that a stochastic mechanism $(X, \Psi, \Phi, \mathcal{F})$ is *eventually weakly anonymous*, if the $k$-clustering mechanism $\Psi$ is eventually weakly anonymous. (Analogous properties can be defined for the other notions of anonymity.)

This definition allows us to consistently formalize the levels of informational isolation that can exist in the economy. As a rule, stronger degrees of anonymity provide stricter restrictions on the informational flows that can take place among agents.

The *eventual weak* anonymity notion captures the idea that the matching mechanisms may allow some agents to repeatedly interact only in the short run. After some period these agents will move out to different clusters. Under *weak* anonymity, instead, clusters cannot be formed with the same agents. It follows that if an agent $a$ is paired to $b$ on some date, then $a$ and $b$ have never met before and will never meet again. However, the possibility exists that $b$ might have met either one of $a$'s past partners or one of $a$'s former clustermates. To remove these possibilities of direct or indirect linkages among agents, we need to add restrictions to the mechanics of matching. These are progressively formalized in the notions of *anonymity* and *strong anonymity*. Strong anonymity removes all possible direct and indirect links among agents who belong to the same cluster. This reflects a suggestion made by Kocherlakota (1998). What's more, strong anonymity rules out any *future* direct and indirect links among these agents and stronger degrees of anonymity imply weaker degrees of anonymity; see Aliprantis et al. (2006, Lemmas 8 and 9).

The central question at this point is whether random pairings exist that are strongly anonymous. That is to say, is it possible to construct a class of matching mechanisms that can ensure total informational isolation in every meeting? We provide an affirmative answer to this challenging question in the next section.

## 5. Constructing anonymous random matches

For anonymous matching mechanisms we need an infinite population $X$. At date $t = 0$, we partition $X$ in a countable number of sets $A_1, A_2, \ldots$ of identical cardinality.[9] Thus, we have an initial partition $X = \bigsqcup_{n=1}^{\infty} A_n$. Then, in each $t \geqslant 1$ we divide $X$ into clusters, building on this initial partition, using $k$ sets at a time. To do this, we need to describe how to partition the population over time. The construction of these partitions—referred to as a *recursive block-*

[9] This means $A_n$ can be countable or uncountable.

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*partition*—is described by the recursive method illustrated below. (The brackets below indicate the partition sets.)

| Period | Block partition of the population $X$ |
|---|---|
| 0 | $X = A_1 \sqcup A_2 \sqcup A_3 \cdots$ |
| 1 | $X = \langle A_1 \sqcup \cdots \sqcup A_k \rangle \sqcup \langle A_{k+1} \sqcup \cdots \sqcup A_{2k} \rangle \sqcup \cdots$ |
| 2 | $X = \langle A_1 \sqcup \cdots \sqcup A_{k^2} \rangle \sqcup \langle A_{k^2+1} \sqcup \cdots \sqcup A_{2k^2} \rangle \sqcup \cdots$ |
| $\vdots$ | $\vdots$ |
| $t$ | $X = \bigsqcup_{n=1}^{\infty} \langle A_{(n-1)k^t+1} \sqcup A_{(n-1)k^t+2} \sqcup \cdots \sqcup A_{nk^t} \rangle$ |

$$= \bigsqcup_{n=1}^{\infty} \bigsqcup_{j=1}^{k^t} A_{(n-1)k^t+j}$$

$$= \bigsqcup_{n=1}^{\infty} B_n^t = \langle B_1^t \sqcup \cdots \sqcup B_k^t \rangle \sqcup \langle B_{k+1}^t \sqcup \cdots \sqcup B_{2k}^t \rangle \sqcup \cdots$$

$$= \bigsqcup_{n=1}^{\infty} \langle B_{kn-(k-1)}^t \sqcup \cdots \sqcup B_{kn}^t \rangle = \bigsqcup_{n=1}^{\infty} B_n^{t+1}$$

$$\vdots$$

where we have defined $B_n^t = \bigsqcup_{j=1}^{k^t} A_{(n-1)k^t+j}$ for $n = 1, 2, \ldots$ and $t \geqslant 1$. For example, for $t = n = 1$ then $B_1^1 = \bigsqcup_{j=1}^{k} A_j = \langle A_1 \sqcup \cdots \sqcup A_k \rangle$.

It should be clear that for each $t \geqslant 1$ the sets $B_1^t, B_2^t, \ldots$ (called the *blocks* of the population in period $t$) are pairwise disjoint and have the same cardinality. Moreover, $B_n^{t+1} = \langle B_{kn-(k-1)}^t \sqcup \cdots \sqcup B_{kn}^t \rangle$ holds for $n = 1, 2 \ldots$ and $t \geqslant 1$, so that $B_n^{t+1}$ is a union of $k$ pairwise disjoint sets of identical cardinality. By Theorem 4, we can construct for each $n$ and $t \geqslant 1$ a $k$-clustering rule $\psi_{n,t} : B_n^{t+1} \twoheadrightarrow B_n^{t+1}$ such that given any $x \in B_n^{t+1}$ the set $\psi_{n,t}(x)$ consists of $k$ agents, one from each of the $k$ blocks $B_{kn-(k-1)}^t, \ldots, B_{kn}^t$. For each $t$ we have a $k$-clustering rule $\psi_t^*: X \twoheadrightarrow X$ defined for each $x \in B_n^{t+1}$ by

$$\psi_t^*(x) = \psi_{n,t}(x).$$

We also let $\psi_0^* = I$, the identity on $X$.

**Definition 16.** Any $k$-clustering mechanism $\Psi^*$ constructed as above is called *recursive block-invariant*.

Since spatial separation guarantees that on each date matches occur among agents that belong to the same cluster, it follows that the recursive block-invariant mechanisms ensure total informational isolation. Here is the result that formalizes this intuition.

**Theorem 17** *(Existence of strong anonymity). Every recursive block-invariant mechanism is strongly anonymous.*

**Proof.** The proof will be based upon the following two properties. For each $n = 1, 2, \ldots$, each $t \geqslant 0$ and each $0 \leqslant \tau \leqslant t$ we have:

(1) $\psi^*_\tau(B^{t+1}_n) = B^{t+1}_n$, and
(2) $\Pi_\tau(x) \subseteq B^{t+1}_n$ for all $x \in B^{t+1}_n$.

The proof of (1) is by induction on $t$. For $t = 0$ it is obvious that $\psi^*_0(B^1_n) = B^1_n$ for all $n$, since by our definition $\psi^*_0(x) = \{x\}$ for all $x \in X$. Therefore, for the induction step, assume that for some $t \geqslant 0$ we have $\psi^*_\tau(B^{t+1}_n) = B^{t+1}_n$ for all $n$ and all $0 \leqslant \tau \leqslant t$. We want to prove that for any given $n$ we have $\psi^*_\tau(B^{t+2}_n) = B^{t+2}_n$ for each $\tau = 0, 1, \ldots, t + 1$. Start by observing that by the induction hypothesis $\psi^*_\tau(B^{t+1}_n) = B^{t+1}_n$ holds true for all $\tau = 0, 1, \ldots, t$. Now note that $B^{t+2}_n = B^{t+1}_{kn-(k-1)} \sqcup \cdots \sqcup B^{t+1}_{kn}$. But then for each $\tau = 0, 1, \ldots, t$ we have

$$\psi^*_\tau\big(B^{t+2}_n\big) = \psi^*_\tau\Bigg(\bigsqcup_{j=kn-(k-1)}^{kn} B^{t+1}_j\Bigg) = \bigcup_{j=kn-(k-1)}^{kn} \psi^*_\tau\big(B^{t+1}_j\big)$$
$$= \bigsqcup_{j=kn-(k-1)}^{kn} B^{t+1}_j = B^{t+2}_n.$$

Also, by definition $\psi^*_{t+1}(B^{t+2}_n) = B^{t+2}_n$. Therefore, $\psi^*_\tau(B^{t+2}_n) = B^{t+2}_n$ holds true for each $n$ and all $\tau = 0, 1, \ldots, t + 1$, and the validity of (1) has been established.

The proof of (2) is by induction on $\tau$. For $\tau = 0$ notice that for each $x \in B^{t+1}_n$ we have $\Pi_0(x) = \{x\} \subseteq B^{t+1}_n$. For the inductive step assume that for some $0 \leqslant \tau < t$ we have $\Pi_\tau(x) \subseteq B^{t+1}_n$ for all $x \in B^{t+1}_n$. We must show that $\Pi_{\tau+1}(x) \subseteq B^{t+1}_n$ for all $x \in B^{t+1}_n$.

Fix $x \in B^{t+1}_n$. From (1) we get $\psi^*_{\tau+1}(B^{t+1}_n) = B^{t+1}_n$, and so $\psi^*_{t+1}(x) \subseteq B^{t+1}_n$. Therefore, each element $y \in \psi^*_{\tau+1}(x)$ belongs to $B^{t+1}_n$. But then our induction hypothesis yields $\Pi_\tau(y) \subseteq B^{t+1}_n$ for each $y \in \psi^*_{\tau+1}(x)$, and so $\Pi_{\tau+1}(x) = \bigcup_{y\in\psi^*_{\tau+1}(x)} \Pi_\tau(y) \subseteq B^{t+1}_n$.

We are now ready to show that $\Psi^*$ is strongly anonymous. To this end, assume that $a, b \in X$ satisfy $a \neq b$, and $b \in \psi^*_{t+1}(a)$ with $t \geqslant 0$. Since $a \in X = \bigsqcup_{n=1}^\infty B^{t+1}_n$ there exists a unique natural number $n$ such that $a \in B^{t+1}_n$. Since the correspondence $\psi^*_{t+1}$ restricted to $B^{t+2}_n$ is $k$-partitional, it follows that there exists some $j \neq n$ such that $b \in B^{t+1}_j$. But then it follows from (2) that $\Pi_t(b) \subseteq B^{t+1}_j$. Using (2) once more we get $\Pi_t(a) \subseteq B^{t+1}_n$. Finally, taking into account that $B^{t+1}_j \cap B^{t+1}_n = \emptyset$ we easily infer that $\Pi_t(a) \cap \Pi_t(b) = \emptyset$, and the proof is finished.[10] □

This theorem demonstrates that (given any infinite population $X$) a simple matching technique exists that ensures complete informational isolation in each match and in each period. The necessary ingredient is an initial partition of the set $X$ composed of countably many pairwise disjoint sets of identical cardinality (see the examples in Aliprantis et al., 2006).

For instance, suppose we want to construct clusters of $k = 3$ agents on a population $X$ consisting of the natural numbers. The initial partition is $X = \bigsqcup_{n=1}^\infty \{n\} = \bigsqcup_{n=1}^\infty A_n$, so each $A_n$ has

[10] Property (1) is related to the notion of invariance with respect to a function. Given a function $f : X \to X$, a subset $S$ of $X$ is said to be $f$-*invariant* if $f(S) \subseteq S$, i.e., $f(x) \in S$ for all $x \in S$. According to this terminology, for each $t \geqslant 0$, each $n$ and all $\tau = 0, 1, \ldots, t$ the sets $B^{t+1}_n$ are $\psi^*_\tau$-invariant. This implies that one can construct strongly anonymous mechanisms as long as $\psi^*_\tau(B^{t+1}_n) \subseteq B^{t+1}_n$. The equality $\psi^*_\tau(B^{t+1}_n) = B^{t+1}_n$ is not necessary for strong anonymity.

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Table 1

| t | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | [1] | [2] | [3] | [4] | [5] | [6] | [7] |
| 1 | [**1**, 2, 3] | [**4**, 5, 6] | [**7**, 8, 9] | [**10**, 11, 12] | [**13**, 14, 15] | [**16**, 17, 18] | [**19**, 20, 21] |
| 2 | [**1**, 4, 7] | [2, 5, 8] | [3, 6, 9] | [**10**, 13, 16] | [11, 14, 17] | [12, 15, 18] | [**19**, 22, 25] |
| 3 | [1, 10, 19] | [2, 11, 20] | [3, 12, 21] | [4, 13, 22] | [5, 14, 23] | [6, 15, 24] | [7, 16, 25] |
| ⋮ | | | | | | | |

cardinality one. According to our recursive block-partition, in $t = 0$ we have $B_n^1 = A_n = \{n\}$. For $t = 1$, we have $B_n^2 = B_{3n-2}^1 \sqcup B_{3n-1}^1 \sqcup B_{3n}^1 = \{3n-2, 3n-1, 3n\}$, and so on. An implementation of the recursive block-invariant mechanism $\Psi^*$ is shown in Table 1.

That is, in $t = 1$, $\psi_1^*(1) = \psi_1^*(2) = \psi_1^*(3) = \{1, 2, 3\}$. In $t = 2$ we have $\psi_2^*(1) = \psi_2^*(4) = \psi_2^*(7) = \{1, 4, 7\}$, and so on. It is easy to see that agents in any cluster have no direct or indirect links to prior clustermates. That is, this mechanism is strongly anonymous.

## 6. An application: random matching models of money

Here we demonstrate how the theoretical construction we have developed can be used to provide a foundation to the existing matching literature. To do so, we focus on search-theoretic models of money. These are models in which infinitely-lived agents are assumed to meet randomly and pairwise over time, but are never paired more than once and cannot observe the trading histories of others. These, as well as additional conditions on preferences and technologies, make trading frictions explicit and provide a definite medium-of-exchange role to fiat money.

The seminal paper in this literature is Kiyotaki and Wright (1989), which describes a discrete-time monetary economy with a continuum population of mass one. Agents can be one of three types, in equal proportions. While the matching technology is not formalized, the paper contains the following description of the outcome of the matching process:

> "..., each period, agents are matched randomly in pairs and must decide whether or not to trade bilaterally, without the benefit of an auctioneer or some other outside authority to impose any arrangement. Trade always entails a one-for-one swap of inventories, given the physical environment, and occurs if and only if mutually agreeable (there is no credit since a given pair will meet again with probability 0)."

We now formalize a matching process that satisfies such a description and explain how to ensure that matching is done so that there is no credit. That is, not only every pair meets again with probability zero (which we called weak anonymity), but *also* we show how to ensure that every pair does not share past partners, etc. In brief, we construct matches in which agents are completely isolated from an informational standpoint.

Here are the steps one needs to take. First, select a population with infinitely many agents. To do so, let for instance $X = \mathbb{N} = \{1, 2, 3, \ldots\}$. Second, divide the population into three types of agents in equal "proportions." Therefore, we let agents $\{1, 4, 7, \ldots\}$ be of type I, agents $\{2, 5, 8, \ldots\}$ be of type II, and agents $\{3, 6, 9, \ldots\}$ be of type III. Third, ensure that each agent has probability $\frac{1}{3}$ to meet an agent of any type in each period. To do so, we restrict attention to $k$-clustering rules which include multiples of three. In this way we can have an equal number of agents of each type in each cluster. Then we choose a probability measure over the set of $\ell_k$ matching rules, such that each agent has probability $\frac{1}{3}$ to be matched to any type.

As an example, let $k = 3$ and consider the cluster $\Omega = \{1, 2, 3\}$. The number of all possible pairings (i.e., bilateral matching rules) is $\ell_3 = 4$. The set that lists all possible bilateral matching rules is $\mathcal{B}(\{1, 2, 3\}) = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\phi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \phi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

If we consider the probability measure $f$ on $\mathcal{B}(\Omega)$ defined by $f(\phi_1) = f(\phi_2) = f(\phi_3) = \frac{1}{3}$ and $f(\phi_4) = 0$, then each type has equal probability of meeting any of the three types.

The number of bilateral matching rules $\ell_k$ grows large very quickly. So, the probability of any given matching rule being chosen drops rapidly to zero as $k$ grows. That is, the chance of meeting any one agent drops to zero rapidly as the size of the cluster grows. To ensure matches are anonymous, we use a recursive block-invariant clustering mechanism. As an illustration, consider agent 1 and the clusters to which he belongs over time as shown in Table 1. In period $t = 0$ agent 1 is by himself. In period $t = 1$ he is in a cluster with agents 2 and 3. In period $t = 2$ agent 1 is in a cluster with agents 4 and 7, and so on. In general, on date $t > 0$ agent 1 is in a cluster with agents $3^{t-1} + 1$ and $2 \times 3^{t-1} + 1$. So, agents are matched randomly and once matched they will never meet again.

## 7. Final remarks

We have formalized a way to model random pairwise interactions, focusing on the links between matching technologies and degrees of informational isolation. Our theoretical construct contributes to building more solid foundations for matching models, can be helpful to study how informational and spatial constraints affect the allocations, and can be used to improve the formulation of spatially separated and informationally isolated trading environments (e.g., see Aliprantis et al., 2005; Boel and Camera, 2006). In fact, our theory shows that, to achieve considerable informational isolation, random matches cannot be "generic" and one has to work hard at constructing anonymous matching mechanisms. One may interpret this feature as saying that substantial anonymity in trading may be perhaps too strong of an assumption.

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