Prisoner's Dilemma Game with Heterogeneous Influential Effect on Regular Small-World Networks

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The effect of heterogeneous influence of different individuals on the maintenance of co-operative behaviour is studied in an evolutionary Prisoner's Dilemma game with players located on the sites of regular small-world networks. The players interacting with their neighbours can either co-operate or defect and update their states by choosing one of the neighbours and adopting its strategy with a probability depending on the payoff difference. The selection of the neighbour obeys a preferential rule: the more influential a neighbour, the larger the probability it is picked. It is found that this simple preferential selection rule can promote continuously the co-operation of the whole population with the strengthening of the disorder of the underlying network.

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Game theory[1−4] was introduced qualitatively to study complex behaviours of biological, ecological, social and economic systems. Of particular renown is the evolutionary prisoner's dilemma game (PDG), which has attracted most attention in theoretical and experimental studies.[3] Recently, much attention has been focused on applications of the PDG in the area of behaviour sciences, biology and economics, etc.[5−7] Intriguingly, the methods developed in statistical mechanics, such as phase transitions, criticality and the concept of universality classes, etc., can be applied to study the spatial evolutionary game theory and has turned out to be very fruitful.[8−10]

In the original PDG the players could make two choices: either to co-operate with their co-players or to defect. They are offered some payoffs dependent on their choices, which can be expressed by 2×2 payoff matrices in agreement with the four possibilities. The players obtain rewards $R(P)$ if both choose to co-operate (defect). If one player co-operates while the other defects, then the co-operator (C) obtains the lowest payoff S (sucker's payoff), while the defector (D) gains the highest payoff T (temptation to defect). Thus the elements of the payoff matrix satisfy the conditions:$T > R > P > S$ and $2R > T + S$, so that leads to the so-called dilemma situation where mutual co-operation is beneficial in a long perspective but egoism can produce big short-term profit.

In studying of the PDG, one of the most interesting items is to study under what conditions the mutual co-operation will emerge and sustain stably or how to facilitate the co-operation of the whole population.[2−4] In the PDG, the state where all players are defectors has proven to be an evolutionary stable state,[11] which has inspired numerous investigations of suitable extensions that enable co-operative behaviour to persist. Nowak and May[11] have introduced a spatial evolutionary PDG model, in which individuals located on a lattice play with their neighbours and with themselves. The dynamics of the game is govern by a deterministic rule: individuals adopt the strategy that has received the highest payoff among its neighbours including themselves in the previous round. It has been shown that the spatial effects promote the survival of co-operators.[11−15] Szabó and Tőke extended the deterministic dynamics of the model to a stochastic evolutionary one: rather than following the most successful neighbour's strategy straightly, the adoption of one of the neighbouring strategies is allowed with a probability dependent on the payoff difference.[8] This revised version took account into the irrational choices of the players and observed that below certain critical values b_c (noise-dependent), a stable absorbing state of all C emerged. Recently the spatial PDG have been studied on different complex network models, it was found that co-operation can be maintained on these networks in a wide range of network parameters.[9,16−20] In addition, the dynamic network model[21] and the dynamic payoff matrices[22] were also introduced to sustain high concentration of co-operators in the evolution of PDG.

In this Letter, we study the PDG using the Szabó–Tőke version[8] on regular small-world networks with slightly different dynamics. The interaction neighbourhood is restricted to the nearest neighbours and no self-interactions are included; rather than randomly selecting a neighbour in comparison,[8,9,20] the players select one of their neighbours to update their states according to a proportional rule (see the definition of the model below). Our main aim is to investigate how the underlying structure of interaction and the dynamics affect the game evolution. Using systematic Monte Carlo (MC) simulations, we calculate the density of co-operators as a function of the temptation to defect b for different disorder levels ϕ of the network. It is found that with the increasing of the disorder of the underlying network and the emergence of heterogeneous influential effect among the players,

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the co-operative behaviour can be enhanced reasonably.

The model and simulation. We consider an evolutionary PDG with players located on regular small-world networks with a ϕ portion of random rewired links, where ϕ characterizes the disorder extent of the network. Figure 1 illustrates an example whose creation is similar to those suggested by Watts and Strogatz.[23] Note that each site has the same fixed number of neighbours. If $\phi = 0.0$, this structure reproduces the square lattice and for $\phi = 1.0$ it is equivalent to a random regular graph.[24] The players are pure strategists and can follow only two simple strategies: C (co-operate) and D (defect). Each player interacts with its neighbours and collects the payoffs depending on the payoff-matrix elements. The total payoff of a certain player is the sum over all interactions. Following the previous studies,[8,11,16,18] the elements of the payoff matrix can be rescaled, i.e. we can choose $R = 1$, $P = S = 0$, and $T = b$ (> 1) without loss of generality in the evolutionary PDG.

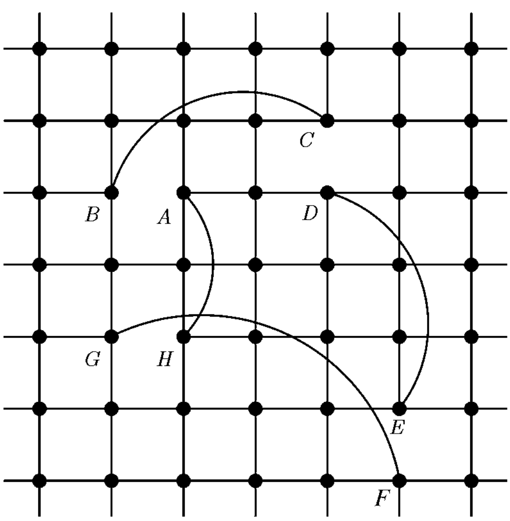

Fig. 1. Structure of a regular small-world network whose construction starts from a square lattice under periodic boundary conditions. First the randomly chosen AB link is removed and the site B is rewired to the randomly chosen site C. To have four connections at site C we eliminate one of the previous links (here CD) and we add a new link DE at random. This process is repeated until ϕ portion of the nearest neighbour bonds are replaced by random links. Finally the last site (here H) is wired to the first one (A). This figure is taken from Ref. [20].

In society, some special persons may influence others much stronger than the average individual, still these influential persons are coupled back to their social surroundings.[18] In other words, different neighbours would have different influence on one's behaviour. In general, one can expect that the influence between two people is asymmetric. To model this situation we define a quantity A_{ij} $(i, j = 1, \ldots, N)$, which describes the influence weight of jth player to ith player and possesses asymmetric property: i.e., independent of the corresponding quantity A_{ji}. For simplicity, we assume that A_{ij} follows a power law distribution $P(A) \sim A^{-\gamma}$ and does not change with time.[25] In this way, we hope to catch some general effects that asymmetric influence among the players might have on the dynamical behaviour of the game.

The randomly chosen player i revises its strategy by selecting one of its neighbours j with a probability Δ according to a proportional rule:

$$\Delta_{ij} = \frac{A_{ij}}{\sum_{k \in \Omega_i} A_{ik}}, \qquad (1)$$

where Ω_i is the community composing of the nearest neighbours of i. Equation (1) means that the larger the influential weight of a neighbour, the more the probability selected to compare with. Accepting the idea suggested by Szabó,[8,9,20] given the total payoffs (E_i and E_j) from the previous round, player i adopts the neighbour's strategy with the probability

$$W = \frac{1}{1 + \exp\left[-(E_j - E_i)/K\right]}, \qquad (2)$$

where E_j is the neighbour's payoff and K characterizes the noise introduced to permit irrational choices. For all the simulation experiment described in the following, the value of K is fixed at 0.1. Generate a random number r uniformly distributed between zero and one, if $r < W$, the neighbour's strategy is imitated.

Two groups of systems will be considered subsequently. In the first case, $\gamma \to \infty$ with $\phi = 0.0, 0.1, 1.0$ is studied. This means that the neighbours of each player will be selected with equal probability to compare with those during the whole process of the evolution. In the second case, $\gamma = 0.8$ with the corresponding ϕ values is investigated: i.e., the influential weight is taken into account. Starting from a random initial state, the rules of the model are iterated with parallel updating by varying the value of b. The total sampling times are 5000 MC steps and all the results shown in the following are averaged over the last 2000 steps.

Results and discussion. In the following we show the results of simulations performed in systems with $N = 100 \times 100$ players. Our key quantity is the co-operator density ρ_C, the average fraction of players adopting the strategy C in the equilibrium state. First we consider the model with random selection case (i.e., the case of $\gamma \to \infty$). For different values of ϕ, we recover qualitatively the results of the stochastic model.[10] From the open symbols in Fig. 2, one can see that ρ_C decreases monotonically with increasing b up to a certain threshold b_c, where the co-operators vanish. With longer-range links emerging on the lattice, the level of co-operation is promoted reasonably, which is different from the previous researches whose results support that the local interaction may promote the co-operation of the whole population (see Refs. [11–13,15] and the references therein). For increasing b, the spatial correlations result in a critical transition on the square lattice ($\phi = 0.0$), whereas on other regular small-world networks ($\phi > 0.0$) the lack of correlations lead to a linear decrease in co-operation, that is, a mean-field type transition.[10]

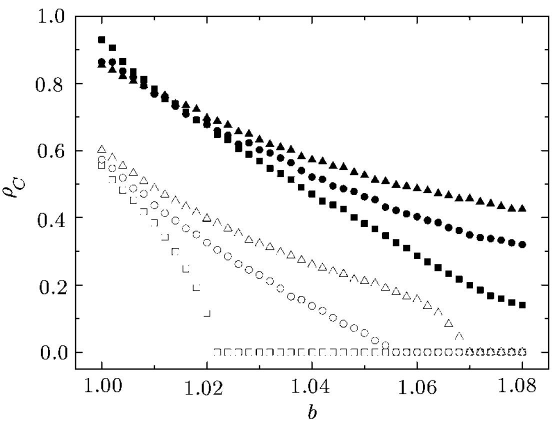

Fig. 2. Average density of co-operators, ρ_c, as a function of the temptation-to-defect b in the equilibrium state. Open and closed symbols correspond to the case of random selection ($\gamma \to \infty$) and preferential selection ($\gamma = 0.8$) of one neighbour to compare with respectively. The different symbols describe different disorder extent of the network: squares for $\phi = 0.0$ (regular lattice); circles for $\phi = 0.1$ (typical regular small-world network), and triangles for $\phi = 1.0$ (regular random network).

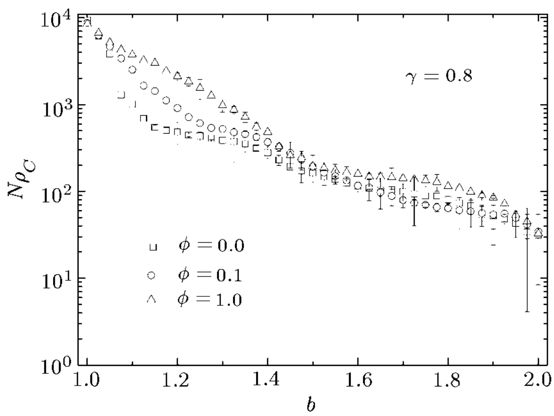

Fig. 3. Cooperator average density ρ_c multiplied by system size N as a function of the temptation-to-defect b in the equilibrium state for $\gamma = 0.8$. To view clearly the minor level of co-operators, the y-axis is plotted by log-scale.

We now consider the influence of the preferential selection on the evolution of the game. The results obtained for $\gamma = 0.8$ are illustrated by the closed symbols in Fig. 2 and in Fig. 3. There are some remarkable differences from the random selection case. There is no absorbing states arising in the whole parameter region $1.0 \leq b \leq 2.0$ (Fig. 3). All co-operators and defectors coexist during the evolutionary process though the same monotonical decreasing trend of ρ_C with the increasing temptation-to-defect. As is indicated, even in the extreme defection circumstance $b \approx 2.0$, co-operators can survive and persist with a minor level. For convenience, the data are plotted with a log-line scale. In Fig. 4, we depict ρ_C as a function of b for different values of γ in the case of $\phi = 0.1$ (the qualitative properties of the result hold for $\phi = 0.0$, and 1.0). There exists an optimal value for $\gamma \approx 0.3$, under which the co-operation is maximally enhanced. For the large values of γ, the co-operation is inhibited (we have found that the result for $\gamma = 2$ is very close to that for the random selection case $\gamma \to \infty$). We conclude that the preferential selection gives rise to the emergence of influential players; and if some of them are co-operators, then compact communities consisting of their neighbours and themselves could be formed and survive stably in the background of defectors, which would contribute to the persistence of the co-operation.

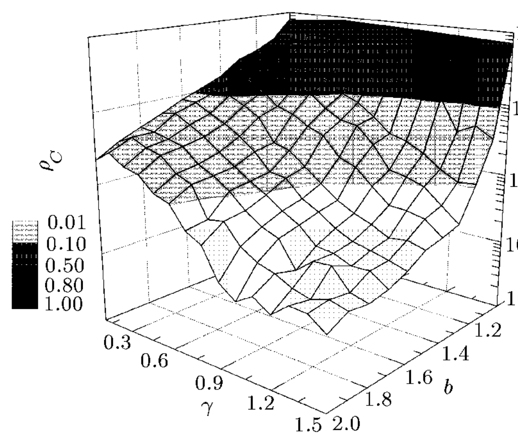

Fig. 4. ρ_c as a function of the temptation to defect b for $\gamma \in [0.1, 1.5]$. The roughness of the front surface is due to the intensive fluctuation of the co-operators under cruel surviving condition (large values of b and γ). The data are averaged over several times, and parameter value $\phi = 0.1$.

In order to check this statement, we also measured the persistence, the number $P(t, t_w)$ of sites of co-operators that do not change strategy between an initial waiting time t_w, and the time $t \geq t_w$.[26] In addition, the correlation function $Q(t, t_w)$, which characterize the number of sites of co-operators at time t that has been arisen at time t_w in spite of finite flipping of the strategy during the two time interval, is also explored. In a distinct view, the results of these two functions for $t_w = 3001$ (this time can be selected arbitrarily as long as the system has attained equilibrium) are summarized in Fig. 5. For the preferential selection case, after an initial decrease, the persistence attains, for large times, a plateau whose value depends both on ϕ and γ. If the persistence does not go to zero, we know that there is a fraction of sites of co-operators that flip only finitely many times (blocking), and domain wall movements are constrained (pinning).[26] For the present system we studied, it is indicated that communities of co-operators exist stably in the background of defectors. However, for the random selection case, the persistence goes to zero in the long-time limit, which means that all the co-operators are re-

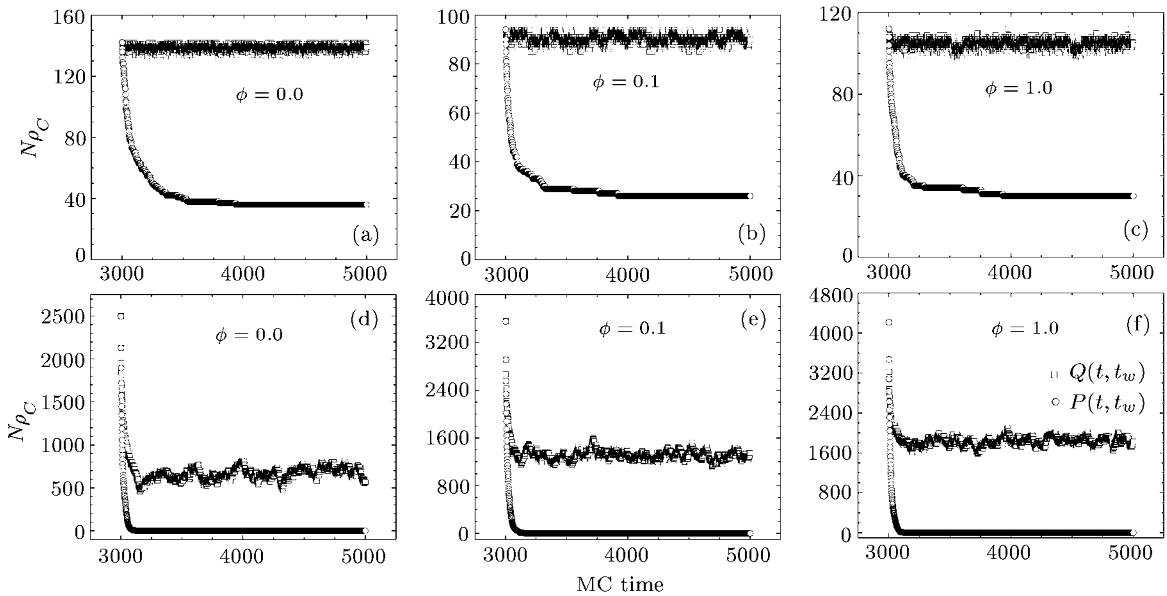

Fig. 5. (a)–(c) preferential selection case for $b = 1.60$; (d)–(e) random selection case for $b = 1.016$. Squares and circles correspond to the persistence function $P(t, t_w)$ and the correlation function $Q(t, t_w)$ of the system in equilibrium state respectively. In the present case, $t_w = 3001$ is selected as an initial waiting time. That the persistence goes to zero in the long-time limit indicates the random walk and annihilation of the co-operators, whereas the plateau value of the persistence implies the stable maintenance of the communities of the co-operators. The same information of the evolution can also be given by the behaviour of the correlation function, which sustains the level of value of the original time in the preferential selection case and displays mean-field behaviour in the random selection case.

newed completely after finite waiting time. This is reminiscent of the random walk and annihilation.[8]

The behaviour of correlation function $Q(t, t_w)$ also gives out the same evolutionary characters of the system. For preferential selection case, this quantity fluctuates weakly around the initial value $Q(t_w, t_w)$, indicating the stable maintenance of the communities of co-operators. For random selection case, the random walk and annihilation of co-operators causes the long-time correlation to be independent of the initial state, which can be calculated roughly by a mean-field approximation method. Since the co-operators can be regarded as walking randomly, the probability of revisiting those sites, which had been visited before, will be in proportion to the average density of the co-operators. Assuming that the number of co-operators is equal to $N(t_w)$ at time t_w (note that the system has already attained equilibrium before this time), the mean-field approximation will give out, $Q(t, t_w) \approx N(t_w) \times N(t_w)/N$, in the long-time limit $t \to \infty$. From Fig. 5, one can find that the analysis is in good agreement with the numerical simulation.

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